[1]Golden section search solution of single variable optimization problems using MathCad®

Eduardo Veras
Assistant Professor, PUPR
Department of Industrial Engineering

Abstract

The method of solution presented in this article is based on a searching technique known as the Golden Section Rule. This method can be used for solving single-variable optimization problems without constraints, but for which variable the initial interval of uncertainty can be specified for searching the optimum values of the design variables. The intention is to facilitate the understanding of how optimization problems can be formulated and solved more easily using existing computer-aided mathematics and calculus packages. MathCad® software is used as the computer tool for this application.

Solución de problemas de optimización de una sola variable por el método de sección aurea usando MathCad®

Sinopsis

El método de solución presentado en esta publicación está basado en la técnica de búsqueda conocida como la Regla de Sección Áurea. Este método se puede usar para resolver problemas de optimización definidos mediante una sola variable de diseño sin restricciones, pero para cuya variable se puede definir un intervalo inicial de incertidumbre de búsqueda de los valores óptimos de las variables de diseño. La intención es facilitar la comprensión de cómo se pueden formular problemas de optimización y cómo se pueden resolver más fácilmente usando sistemas de

®MathCad is a registered trademark of MathSoft, Inc.

computadoras pre-programados disponibles para matemáticas y cálculo. El "software" conocido como MathCad® se usó como la herramienta de computadora para esta aplicación.

Introduction

Optimization is a mathematical formulation for finding the best possible solution to a problem that may include limiting constraints. Graphically, an optimization problem can be visualized as finding the lowest or highest points in a complex and contoured landscape. An optimization algorithm searches for the topological extremes (usually called critical points) within a set of feasible solutions that is, those which satisfy the limiting constraints.

The mathematical formulation of an optimization problem starts by defining the design variables. Most of the time the values these variables can take within the solution space are limited by the constraints imposed on the design of the physical system. For example, if we consider the design of a truss bar for a bridge, we must consider the physical properties of the material to resist the applied load, speed of the wind, geometry of the cross-sectional area of the bar, member size, weight of the bridge itself, budget allotted for the construction and other considerations. There can be several feasible designs for this system, but some are better than others. A criterion is needed to decide whether an obtained solution is better than another. This criterion is called the objective function for the optimization problem.

There are many mathematical methods for solving optimization problems. Basically, they can be divided into two broad groups: methods based on gradients (differential calculus) and methods based on searching techniques. These methods may include constraints, in which case it is called a constrained optimization method, or may not include constraints, in which case it is called unconstrained optimization method.

The method of solution presented in this article is based on a searching technique known as the golden section rule. This method can be used for solving single variable optimization problems without constraints, but for which problem we can specify the range values of the design variables within the solution.

The intention is to facilitate the understanding of how an optimization problem can be formulated and solved more easily by using existing computer-aided mathematics and calculus packages. In this particular case, the MathCad® software is used. Optimization is a broad topic and it is not the purpose of this paper to provide an extensive coverage of the existing methodologies, but rather a brief introduction to a non sophisticated solution for students and others at an introductory level in this field.

Mathematical formulation of an optimization problem

The general format of an optimization problem is defined as finding a n-vector $\{x\} = \{x_1, x_2, x_3, \ldots, x_n\}$ of the design variables to minimize or maximize an objective function of the form :

$$f(x) = f(x_1, x_2, x_3, \ldots, x_n) \quad (1)$$

and is subjected to p equality constraints, generally expressed as:

$$h_j(x) = h_j(x_1, x_2, x_3, \ldots x_n) = 0$$
$$j = 1 \ldots p \quad (2)$$

and subjected to m inequality constraints, expressed as:

$$g_i(x) = g_i(x_1, x_2, x_3, \ldots, x_n) \leq 0 \; i = 1 \ldots m \quad (3)$$

where p is the total number of equality constraints and m is the total number of inequality constraints. When "≥ type" inequality constraints are present, these can be easily converted to "≤ type" inequality constraints by multiplying by -1 and the general format is maintained.

Golden section search method

The golden section search method is based on the golden section rule, which consists in dividing the search interval, R, into two unequal intervals such that the ratio of the smaller, R2, to the larger part of the interval, R1, is equal to the ratio of the larger, R1, to the whole, R.

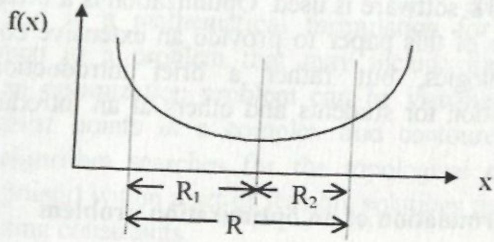

Figure 1. Golden section intervals definition

As figure 1 shows, the golden section rule states that:

$$\frac{R_2}{R_1} = \frac{R_1}{R} \qquad (4)$$

where,

$$R = R_1 + R_2 \qquad (5)$$

Combining equations (4) and (5) and simplifying, the following expression is obtained:

$$\left(\frac{R_2}{R_1}\right)^2 + \left(\frac{R_2}{R_1}\right) = 1 \qquad (6)$$

Denoting $\bar{R} = \left(\frac{R_2}{R_1}\right)$, equation (6) can be expressed as:

$$\bar{R}^2 + \bar{R} - 1 = 0 \qquad (7)$$

Equation (7) is a nonlinear algebraic equation of second order which has two solutions. However, the golden section method uses the positive solution, that is, $\overline{R} = 0.6180$. The direction of search for obtaining the optimal solution is established by repeated application of the golden section rule, that is, subdividing the interval of search by the value of $\overline{R} = 0.6180$.

The golden section search method is applicable if the objective function, f(x) is unimodal within the interval of search. This means that the given function, f(x), must have a single maximum or minimum within the interval of search, R.

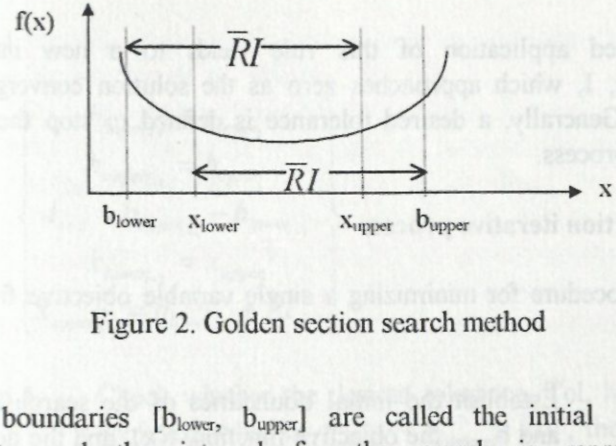

Figure 2. Golden section search method

The boundaries [b_{lower}, b_{upper}] are called the initial interval of uncertainty and the interval for the search is represented as $I = b_{upper} - b_{lower}$. If the points x_{lower} and x_{upper} are defined within the interval [b_{lower}, b_{upper}], as figure 2 shows, and the golden section rule is followed, the point x_{upper} must be placed such that:

$$x_{upper} = b_{lower} + \overline{R}I \tag{8}$$

and the point x_{lower} must be placed such that:

$$x_{lower} = b_{upper} - \overline{R}I \tag{9}$$

Veras/Golden section search solution

The golden section rule ensures that the direction of search when searching for an optimum value to the right of x_{lower} or to the left of x_{upper} is such that the interval of uncertainty is the same from either end of the interval of search.

The objective function, f(x), must be evaluated at these two points and the values obtained compared: for minimization of the objective function, if $f(x_{lower}) \succ f(x_{upper})$, the search is confined to the interval $\left[x_{lower}, b_{upper}\right]$; otherwise, the search is confined to the interval $\left[b_{lower}, x_{upper}\right]$.

Repeated application of this rule yields to a new interval of uncertainty, I, which approaches zero as the solution converges to an optimum. Generally, a desired tolerance is defined to stop the iterative searching process.

Golden section iterative process

The procedure for minimizing a single variable objective function is as follows:

Step 1. Establish the initial boundaries of the search, b_{lower} and b_{upper}, the objective function, f(x), and the desired tolerance, Tol.

Step 2. Calculate the initial values for evaluating the objective function, x_{lower} and x_{upper}, using equations (8) and (9).

Step 3. Compare the values of the objective function, f(x), evaluated at points x_{lower} and x_{upper}. If,

$$f(x_{lower}) \le f(x_{upper}) \begin{cases} b_{lower_{i+1}} = b_{lower_i} \\ b_{upper_{i+1}} = x_{upper_i} \\ I_{i+1} = b_{upper_{i+1}} - b_{lower_{i+1}} \\ x_{lower_{i+1}} = b_{upper_{i+1}} - RI_{i+1} \\ x_{upper_{i+1}} = x_{lower_i} \end{cases} \qquad (10)$$

After updating the points, go to step 5. Otherwise, update the points as in step 4.

Step 4.

$$\begin{cases} b_{lower_{i+1}} = x_{lower_i} \\ b_{upper_{i+1}} = b_{upper_i} \\ I_{i+1} = b_{upper_{i+1}} - b_{lower_{i+1}} \\ x_{lower_{i+1}} = x_{upper_i} \\ x_{upper_{i+1}} = b_{lower_{i+1}} + RI_{i+1} \end{cases} \qquad (11)$$

Step 5. Check whether the desired tolerance, Tol, has been achieved, whether $b_{upper_i} - b_{lower_i} \le Tol$, then go to step 6. Otherwise, set $i = i+1$, and return to step 3.

Step 6. Output the value of x that makes the objective function, f(x), to have a minimum value:

$$f(x)_{\min} = \min\left(f(x_{lower_i}), f(x_{upper_i})\right) \qquad (12)$$

and

$$x_{optimum} = \left(x_{lower}, x_{upper}\right) \qquad (13)$$

Veras/Golden section search solution

The golden section optimization implementation using MathCad® presented is based on the preceding steps.

Implementation

The implementation of the golden section search method is performed through the optimization of a nonlinear objective function,

$$f(x) = 1 - x\sin(\pi x^2) \tag{14}$$

in the interval [0.25,1] and using a desired tolerance, Tol, equal to 0.0001.

MathCad® software is a powerful and general-purpose computer-aided mathematics and calculus package. It is particularly useful for implementing matrix algebra, conditionals and iterative processing of mathematical expressions. It also provides a symbolic processor for evaluation of symbolic mathematical operations. Appendix listing corresponds to the MathCad® implementation of the golden section search method for solving the proposed objective function

$$f(x) = 1 - x\sin(\pi x^2) \tag{15}$$

Conclusion

In this implementation of the golden section search method a nonlinear objective function was optimized following the golden section rule for dividing the initial interval of search. The optimum solution was found at x=0.739. The application developed shows a good agreement with the solution obtained using the MathCad® built-in functions, min(). With numerical modeling implemented, various initial intervals of uncertainty and their effect on the convergence of the procedure can be investigated. This approach provides a better understanding of the iterative nature of optimization methods based on search algorithms.

References

Arora, J., 1989, *Introduction to Optimum Design*, McGraw-Hill, New York

Onwubiko, C., 1989, *Foundations of Computer Aided Design*, West Publishing Company, New York

Veras, E., 1994, "Computer Aided Design and Computer Aided Manufacturing," CAD/CAM, Course Notes ME-543, Mechanical Engineering Department, Polytechnic University of Puerto Rico, Puerto Rico

Veras/Golden section search solution

Appendix

$N := 10$ GolderSearch Solution of a Single Variable Minimization Problem

$i := 0..N$

$bu_0 := 1.0$ Upper boundary

$bl_0 := 0.25$ Lover boundary

$Tol_i := 0.0001$ Tolerance

Initial variable settings for the Golden Section Search

$I_0 := bu_0 - bl_0$ Initial interval of uncertainty

$R := 0.618$ Reduction value of the interval of uncertainty = Golden value

$xu_0 := bl_0 + I_0 \cdot R$

$xl_0 := bu_0 - I_0 \cdot R$

$$\begin{bmatrix} bl_{i+1} \\ bu_{i+1} \\ I_{i+1} \\ xl_{i+1} \\ xu_{i+1} \\ funcl_i \\ funcu_i \end{bmatrix} := \text{until}\left[I_i - Tol_i, \text{if}\left[1 - xl_i \cdot \sin\left[\pi \cdot (xl_i)^2\right] \le 1 - xu_i \sin\left[\pi \cdot (xu_i)^2\right], \begin{bmatrix} bl_i \\ xu_i \\ bu_i - bl_i \\ bu_i - R I_i \\ xl_i \\ 1 - xl_i \sin\left[\pi \cdot (xl_i)^2\right] \\ 1 - xu_i \sin\left[\pi \cdot (xu_i)^2\right] \end{bmatrix}, \begin{bmatrix} xl_i \\ bu_i \\ bu_i - bl_i \\ xu_i \\ bl_i + R I_i \\ 1 - xl_i \sin\left[\pi \cdot (xl_i)^2\right] \\ 1 - xu_i \sin\left[\pi \cdot (xu_i)^2\right] \end{bmatrix} \right]\right]$$

$xl^T = (0.536\ 0.714\ 0.536\ 0.714\ 0.604\ 0.714\ 0.646\ 0.714\ 0.739)$

$xu^T = (0.714\ 0.714\ 0.714\ 0.823\ 0.714\ 0.646\ 0.714\ 0.739\ 0.672)$

$funcl^T = (0.578\ 0.287\ 0.578\ 0.287\ 0.449\ 0.287\ 0.376\ 0.287)$

$funcu^T = (0.287\ 0.287\ 0.287\ 0.301\ 0.287\ 0.376\ 0.287\ 0.269)$