Classification of nr-graphs

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Abstract

When all possible *r*-gons are inscribed in *an n*-gon, the resulting *nr*-graph falls into one of the following three categories: *s*-regular (*sr* or strongly regular), μ-regular, or regular (*k*-regular). The definitions and necessary conditions to find the parameters, including the study of such graphs in terms of completeness, are developed throughout the work.

Clasificación de gráficas nr

Sinopsis

Cuando todos los posibles *r*-ágonos se inscriben en un *n*-ágono, la gráfica *nr* resultante corresponde a una de las tres categorías siguientes: *f*-regular (*fr* o fuertemente regular), μ-regular, o regular (*k*-regular). Las definiciones y condiciones necesarias para determinar los parámetros, incluyendo el estudio de tales gráficas en términos de su completitud, se desarrollan a través del trabajo.

Definition 1.

A *graph* is a pair $(G,\perp)$ where G is a finite non-empty set, and $\perp$ a relation in G, called *adjacency* or *orthogonality*, satisfying:

1. $(v_1,v_2)\in\perp \Rightarrow (v_2,v_1)\in\perp$ (symmetric)

2. $(v,v)\notin\perp$, $\forall v\in G$ (antireflexive)

Definition 2.

An *isomorphism of graphs*, $f(X,\perp) \to (Y,\perp')$ is a bijection $f : X \to Y$ such that $x \perp x'$ if and only if $f(x) \perp' f(x')$.

Remark 1.

i) The elements of G are called *vertices*, and those of $\perp$ *edges* (*lines*).

ii If $(v_1,v_2) \in \perp$ (usually denoted $v_1 \perp v_2$) we say that v_1 and v_2 are *adjacent*.

Definition 3.

An *nr-graph*, *nr*-G, is a pair (G, $\perp$) where G is a finite non-empty set, and $\perp$ a relation in G, called adjacency or orthogonality, satisfying:

1. $(v_i,v_j) \in \perp \ \forall j \ni \ i+1 \leq j \leq n-2r+i+2, \ n \geq 2r, \ n,r,i,j \in N$ (natural numbers), $i \neq j$
2. $v_k = v_{k(\mathrm{mod}\, n)} \ \forall k \in N$.

Remark 2.

In an *nr*-graph, *nr*-G, the adjacency relation can be equivalently restated as follows: $(v_i,v_j) \in \perp \ \forall j \ni \ 2r+k-2 \leq j \leq n+k-1$. In fact, if we let $i=2r+k-3$ in *Definition 3*, we have $(v_i,v_j) \in \perp \ \forall j \ni \ (2r+k-3)+1 \leq j \leq n-2r+(2r+k-3)-2$, or $2r+k-2 \leq j \leq n+k-1$.

Remark 3.

From *Definition 3*, the relation $\perp$ is antireflexive, and from Remark 2, $\perp$ is symmetric. Therefore, according to *Definition 1*, an *nr*-graph is a graph.

Definition 4.

A polygon of *r* sides, or *r*-gon, A, is said to be inscribed in a polygon of *n* sides, or *n*-gon, B, if the vertices of A are non-consecutive vertices of B.

Definition 5.

An *nr*-gon, *nr*-P, is an *n*-gon with all possible inscribed *r*-gons [1].

Definition 6.

An *oriented r-gon* is a polygon of *r* sides obtained by reading, writing, or sketching its vertices and sides, either clockwise or counterclockwise.

Note 1. Throughout this work, the symbol $\prec$ will be used to indicate *precedence*, and the symbols $\uparrow$ and $\downarrow$ to indicate counterclockwise and clockwise directions respectively.

Theorem 1.

$\forall n>5$ nr-graphs, nr-G, are isomorphic to nr-gons, nr-P.

Proof:

Let $\{v_i : 1 \le i \le n, n > 5, v_i \perp v_{i+1}, v_k \equiv v_{k(\mathrm{mod}\,n)}\ \forall k \in N\}$ correspond to the vertices of a $\downarrow$ oriented *n*-gon. The adjacency relation in the *nr*-gon can now be determined as follows: For a given vertex v_i, $v_j \perp v_i$ if $j \neq i$ and, writing $|\{...\}|$ for "cardinality of", $|\{v_k : i \prec k \prec j \uparrow \vee \downarrow\}| \ge 2r-3$.

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[1] J. Sarmiento, 1987, "A Class of Strongly Regular Graphs", LBL-23693, Lawrence Berkeley Laboratory/UC, Berkeley, CA, 94720

:$\Rightarrow$ If $v_i \perp v_j$ in *nr*-G, then $i+1 \leq j \leq n-2r+i+2$, or $i+1 \leq j \leq n-(2r-3)+i-1$ and therefore $|\{v_k : i \prec k \prec j \uparrow\}| \geq 2r-3$. On the other hand, if $2r+i-2 \leq j \leq n+i-1$, $|\{v_k : i \prec k \prec j \downarrow\}| \geq 2r-3$. Notice that if $i=1$, $2r-1 \leq j \leq n$. Therefore, $\uparrow$, between v_{2r-1} and v_1 there are $(2r-1)-2 = 2r-3$ vertices.

$\Leftarrow$: If $|\{v_k : i \prec k \prec j \uparrow\}| \geq 2r-3$ then $i+1 \leq j \leq n-(2r-3)+i-1$ or $i+1 \leq j \leq n-2r+i-2$, and if $|\{v_k : i \prec k \prec j \downarrow\}| \geq 2r-3$ then $(2r-3)+1+i \leq j \leq n+(i-1)$ or $(2r-1)+(i-1) \leq j \leq n+(i-1)$.

Definition 7.

The *degree*, k, of a vertex is the number of vertices adjacent to it.

Definition 8.

A graph is *strongly regular* with parameters k, λ, μ, in symbols G(k, λ, μ), if [2]

- The degree of each vertex is k
- Any two adjacent vertices are mutually adjacent to λ others
- Any two non-adjacent vertices are mutually adjacent to μ others.

Definition 9.

A graph is *μ- regular* with parameters k, μ, in symbols G(k, μ), if

- The degree of each vertex is k
- Any two non-adjacent vertices are mutually adjacent to μ others.

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[2] J. Sarmiento, 1996, "Inscribed r-gons: A Combinatorial Approach", Polytechnic University of Puerto Rico, Vol. 6, Num. 2.

Definition 10.

A graph is *k- regular*, or simply *regular*, in symbols G(k), if the degree of each vertex is k.

Definition 11.

An *nr*-graph is *complete* if $v_i \perp v_j$ $\forall j \neq i$.

Definition 12.

An *nr*-graph is *quasi-complete*, or *q-complete*, if n is even, and $v_i \perp v_j$ $\forall j \neq i$, $j \neq n/2 + i$.

Theorem 2.

An *nr-graph is complete if and only if* $n \geq 4r-5$.

Proof.

$\Rightarrow$ By contradiction. Suppose $n<4r-5$. Then $n+1 < 4r-4$, $n+1-2r < 2r-4$, $n+1-2r+3 < 2r-1$, and $n+1-(2r-3) < 2r-1$. From the definition of *nr*-graph, $v_1 \perp v_j$, $2 \leq j \leq n-(2r-3)$, and from *Remark 2*, $v_1 \perp v_j$, $2r-1 \leq j \leq n$. Since $n+1-(2r-3) < 2r-1$, $(2r-1)-[n-(2r-3)] > 1$, which means that there is at least one j such that $n-(2r-3) < j < 2r-1$ for which $(v_1,v_j) \notin \perp$.

$\Leftarrow$: If $n \geq 4r-5$, then $n+1 \geq 4r-4$, $n+1-2r+2 \geq 2r-2$, $n-(2r-3) \geq 2r-2$, $n-(2r-3) \geq (2r-1)-1$, and $n-(2r-3)+(i-1) \geq (2r-1)-1+(i-1)$.

If $n-(2r-3)+(i-1) = (2r-1)-1+(i-1)$, it follows from the definition of *nr*-graph and *Remark 2* that $v_i \perp v_j$, $i+1 \leq j \leq n+i-1$, without overlaping edges. If $n-(2r-3)+(i-1) > (2r-1)-1+(i-1)$, then $n-(2r-3)+(i-1) > (2r-1)+(i-1)$ and $v_i \perp v_j$, $i+1 \leq j \leq n+i-1$ with at least one overlaping edge. In both instances, $v_i \perp v_j$ $\forall j \neq i$.

Theorem 3.

An *nr-graph is q-complete if and only if n=4r-6.*

Proof:

The necesary and sufficient condition for an *nr*-graph to be *q*-complete is that $v_i \perp v_j$ $\forall j \neq n/2 + 1$, n even, which is equivalent to say that n-(2r-3)+(i-1)+2 = (2r-1)+(i-1) or n-25+5 = 2r-1, that is n=4r-6.

Definition 13.

$\Diamond_I \uparrow = \{v_k : k = (i+2r-2)(\text{mod } n), \ldots, (n+i-1)(\text{mod } n)\}$ and $\Diamond_i \downarrow = \{v_k : k = (i+1)(\text{mod } n), \ldots, (i+n-2r+2)(\text{mod } n)\}$ are the *orbits* of v_i, counterclockwise and clockwise respectively.

Definition 14.

$\lfloor m \rfloor$ is the greatest integer $\leq$ m.

Theorem 4.

i) An nr-graph is strongly regular, with parameters $k=n-1$, $\lambda=n-2$, *and* $\mu=0$, *if* $3 \leq r \leq \lfloor (n+5)/4 \rfloor$, *or with parameters* $k=\mu=n-2$, *and* $\lambda=n-4$, *if* $r=(n+6)/4=\lfloor (n+5)/4 \rfloor + 1$.

ii) If $r > \lfloor (n+5)/4 \rfloor$, $r \neq (n+6)/4$, *and* $I = \Diamond_1 \uparrow \cap \Diamond_{n-2r+5} \downarrow \neq \varnothing$, *then the nr-graph is μ-regular with parameters* $k=2n-4r+4$, $\lambda=0$, *and* $\mu=(n-2r+2)+|\{n, n-1, \ldots, 2r-1\} \cap \{n-2r+5, n-2r+6, \ldots, 2(n-2r+3)\}|$.

iii) If $r > \lfloor (n+5)/4 \rfloor$, $r \neq (n+6)/4$, *and* $I = \Diamond_1 \uparrow \cap \Diamond_{n-2r+5} \downarrow = \varnothing$, *then the nr-graph is k-regular, or simply regular, with* $k=2n-4r+4$.

Proof:

i) $3 \leq r \leq \lfloor (n+5)/4 \rfloor \Rightarrow r \geq 3$ and $4r \leq n+5$ or $n \geq 4r-5$. From *Theorem 2*, the *nr*-graph is complete, and

- $v_i \perp v_j\ \forall j \neq i \Rightarrow$ the degree of each vertex is $k=n-1$
- $v_i \perp v_j \Rightarrow (v_i, v_j) \perp v_k\ \forall k \neq i,j \Rightarrow \lambda=n-2$
- By default $\mu=0$.

Therefore the *nr*-graph is strongly regular.

$r=(n+6)/4=\lfloor (n+5)/4 \rfloor+1 \Rightarrow 4r=n+6$ or $n=4r-6$. From *Theorem 3*, the *nr*-graph is *q*-complete, and

- $v_i \perp v_j\ \forall j \neq i,\ n/2+i \Rightarrow k=n-2$
- $v_i \perp v_j \Rightarrow (v_i, v_j) \perp v_k\ \forall k \neq i, j, n/2+i, n/2+j \Rightarrow \lambda=n-4$
- $(v_i, v_{n/2+i}) \perp v_k\ \ \forall k \neq i, n/2+i \Rightarrow \mu=n-2.$

Therefore the *nr*-graph is strongly regular.

ii) $r > \lfloor (n+5)/4 \rfloor \Rightarrow 4r > n+5 \Rightarrow n < 4r-5$. Thus, by *Theorem 2*, the *nr*-graph is not complete.

$r \neq (n+6)/4 \Rightarrow n \neq 4r-6$. Thus, by *Theorem 3*, the *nr*-graph is not q-complete.

From the definition of *nr*-graph, it follows that each vertex is adjacent to $2[n-(2r-3)-1]$ other vertices, excluding itself. Therefore the *nr*-graph is *k*-regular with $k=2n-4r+4$.

Since the graph is k-regular, to check μ regularity it is sufficient to study the common adjacencies of a particular vertex, let us say v_1, and its non-adjacents, being the first $v_{n-(2r-3)+1}$ or v_{n-2r+4}.

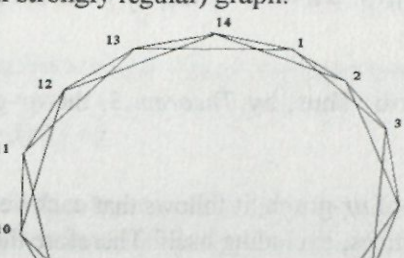
In general, each vertex, v_i, is adjacent to n-(2r-3)-1=n-2r+2 vertices, v_k, on each side, i.e., clockwise and counterclockwise respectively. In particular, the adjacencies of v_1 are $[v_1]=\{v_k : k=2, 3, ..., n-2r+3; n, n-1, ..., n-(n-2r+1)=2r-1\}$.

Similarly $[v_{n-2r+4}]=\{v_k : k=n-2r+3, n-2r+2, ..., (n-2r+4)-(n-2r+2)=2; n-2r+5, n-2r+6, ..., (n-2r+4)+(n-2r+2)=2(n-2r+3)\}$. Thus, $\mu_{1,n-2r+4}=|[v_1]\cap[v_{n-2r+4}]|=|\hat{v}_1 \downarrow \cap \hat{v}_{n-2r+4} \uparrow|+|\{n, n-1, ..., 2r-1\}\cap\{n-2r+5, n-2r+6, ..., 2(n-2r+3)\}|=$ $n-2r+3-1+|\{n, n-1, ..., 2r-1\}\cap\{n-2r+5, n-2r+6, ..., 2(n-2r+3)\}|$ (1).

Now, $I=\hat{v}_1 \uparrow \cap \hat{v}_{n-2r+5} \downarrow \neq \emptyset$ implies that the common adjacency lost clockwise when we go from $v_1 v_{n-2r+4}$ to $v_1 v_{n-2r+5}$, is gained counterclockwise. Therefore $\mu_{1,n-2r+5}=\mu_{1,n-2r+4}$. From the regularity of the graph we can conclude that μ is constant and equal to (1).

iii) If $I=\emptyset$ then $\mu_{1,n-2r+5}=\mu_{1,n-2r+4}-1$, in which case the *nr*-graph is simply *k*-regular with k=2n-4r+4.

For example, figure 1 is a graphical illustration of a regular, figure 2 shows a μ-regular, and figure 3 shows a q-complete, and figure 4 a shows a complete (both strongly regular) graph.



k- regular 14,7-G(4)

Figure 1. Illustration of a k-regular graph

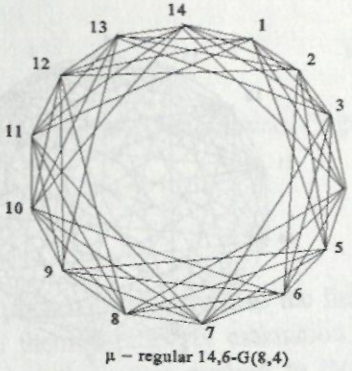


Figure 2. Illustration of a μ-regular graph

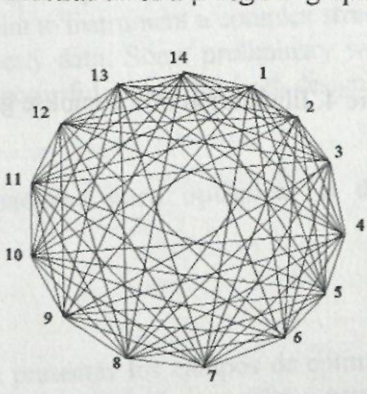


Figure 3. Illustration of a q-complete graph

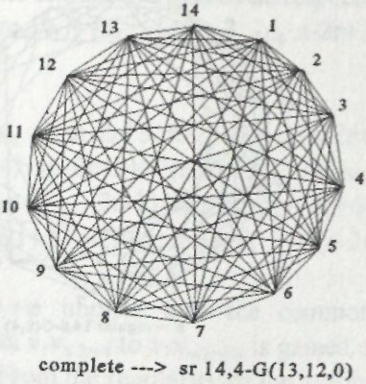


Figure 4. Illustration of a complete graph