Hyperbolic, Ultrastatic Wormhole Solution to the Vacuum Effective Gravity Equations in Randall Sundrum Brane World

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ABSTRACT

We provide a traversable wormhole solution to the effective vacuum gravity equations in Randall Sundrum brane world scenario. We argue that graviton effects from the bulk spacetime can stabilize the wormhole's throat against collapse without the need of any "exotic matter" on the brane. We also construct an ultrastatic, spherically symmetric, horizon free wormhole with spatial hyper surfaces of $\mathbb{R}\times S^2$ topology. The components of the effective energy momentum tensor, induced by the Weyl five dimensional curvature tensor, were calculated explicitly as functions of the radial proper distance and the radius of the throat of the wormhole.

SINOPSIS

Se presenta una solución a las ecuaciones de vacío gravitacionales en el modelo Randall Sundrum para un agujero de gusano transitable. Argumentamos que efectos asociados a estados gravitones que se propagan en el espacio de cinco dimensiones podrían estabilizar la garganta del agujero contra el colapso, sin la necesidad de masa exótica en la membrana que corresponde a nuestro universo. Se construye una solución ultra estática, de simetría esférica, libre de horizonte, con secciones espaciales de topología hiperbólica $\mathbb{R}\times S^2$. Los componentes del tensor efectivo de masa-energía y del tensor de curvatura de Weyl en cinco dimensiones se calculan explícitamente como funciones de la distancia radial y el radio de la garganta.

Key Words: *brane world, wormholes, M-theory*

I- INTRODUCTION

M-Theory is the leading candidate for the theory of everything, describing all the fundamental forces of nature. Horawa and Witten [1] have shown that heterotic string theory with gauge group $E^8\times E^8$ can be realized as eleven dimensional M-Theory compactified to $M^4\left(\frac{S_1}{Z_2}K^6\right)$, where K_6 is a compact Calabi-Yau manifold, and $\frac{S_1}{Z_2}$ is an orientifold with hyper planes at the fixed points of Z_2 symmetry [2]. Presumably the Calabi-Yau space forms a very small structure, suggesting that our universe can be seen as a four dimensional boundary of five dimensional spacetime. This perspective of cosmology was pioneered by Randall and Sundrum and is known as brane world cosmology [3].

In this paper we study a very simple kind of traversable wormhole in the context of brane world. Wormholes are defined as any compact region of spacetime with a topologically simple boundary but a topologically non trivial interior. Essentially wormholes are tunnels connecting distant regions of spacetime or even parallel universes. It has been argued that quantum gravity may allow fluctuations of the topology of the spacetime manifold, but is generally believed that the topology, of causally well behaved spacetimes, do not change as a function of time. In order to avoid topology change theorems, we are forced to consider permanent wormholes of the Lorentzian variety. Euclidian wormholes are usually considered as O(4) instantons in the gravity field and therefore "transient" in nature [4]. Morris and Thorne [5] in their classical paper on traversable wormholes studied the properties of the energy momentum tensor that will keep a non rotating, horizon free traversable wormhole from collapsing. They found that matter in the wormhole spacetime must violate the null energy condition [5]. Although small violations of the null energy condition are observed at the quantum level, large enough violations to sustain a traversable wormhole are very unprovable. In this paper we explore the possibility of a traversable wormhole maintained by the gravity effects of the five dimensional bulk spacetime, without any need of exotic matter on our brane universe. First we present a very short introduction to the effective gravity field equations in the brane following Kei-Chi Maeda [6], then the formalism is applied to a Lorentzian, non rotating, spherically symmetric and horizon free wormhole spacetime with spatial hypersurfaces of $\mathbb{R}\times S^2$ topology.

II- EFFECTIVE GRAVITY FIELD EQUATIONS

Following Randall and Sundrum we consider a five dimensional anti De-Sitter spacetime (M, g_{AB}) with a negative cosmological constant ${}^{(5)}\Lambda<0$ as our model for the bulk spacetime, and a four dimensional hyperplane $(M, g_{\mu\nu})$ with a positive tension $\lambda>0$, located in the hypersurface $B(X_A)=0$ as the brane.

The effective gravity field equations on the brane are given by

$$G_{\mu\nu}=-\Lambda g_{\mu\nu}+8\pi G_N T_{\mu\nu}+\kappa_4^5\Pi_{\mu\nu}-E_{\mu\nu} \quad (1.1)$$

$$\Lambda=\frac{1}{2}\left({}^{(5)}\Lambda+\frac{1}{6}\kappa_5^4\lambda^2\right), \quad (1.2)$$

$$G_N=\frac{\kappa_5^4\lambda}{48\pi}, \quad (1.3)$$

$$\Pi_{\mu\nu}=-\frac{1}{4}T_{\mu\alpha}T_\nu^\alpha+\frac{1}{12}TT_{\mu\nu}+\frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta}-\frac{1}{24}g_{\mu\nu}T \quad (1.4)$$

where

${}^{(5)}\Lambda$ - cosmological constant of the bulk,

Λ - cosmological constant of the brane,

λ - brane tension,

κ_5 - five dimensional gravitational constant,

G_N - Newton's gravitational constant,

$T_{\mu\nu}$ - energy momentum tensor for matter confined to the brane.

The conditions on the brane tension λ and the bulk's cosmological constant ${}^{(5)}\Lambda$ are needed to confine matter and gravity to the brane [3]. Conservation of matter requires

$$T_\mu^\mu=0 \quad (1.5)$$

The tensor $E_{\mu\nu}$ is called the residual 5 dimensional Weyl curvature tensor, and it described local and non local gravitational effects on the brane due to gravitons (closed string states) propagating in the bulk spacetime. The effect of closed strings graviton states being able to propagate through bulk spacetime, while matter is confined to the brane, provides and explanation to the weakness of the gravity force compared to the other forces of nature [7]. The weyl five dimensional curvature tensor is traceless $(E_\mu^\mu=0)$, and satisfies the following constraint equation

$$D^\nu E_{\mu\nu}=\kappa_5^4 D^\nu\Pi_{\mu\nu}$$
$$=\frac{1}{4}\kappa_5^4\left[T^{\alpha\beta}\left(D_\mu T_{\alpha\beta}-D_\beta T_{\mu\alpha}\right)+\frac{1}{3}\left(T_{\mu\nu}-g_{\mu\nu}T\right)D^\nu T\right] \quad (1.6)$$

Unfortunately these equations alone do not form a closed system so, in general the bulk's gravitational field equations have to be solved simultaneously with the brane's effective equations [6].

In this paper we are interested in the vacuum field equations with cero brane cosmological constant. This can be achieved by setting $T_{\mu\nu}=0$, and

$$^{(5)}\Lambda = -\frac{1}{6}\kappa_5^4\lambda^2 \qquad (1.7)$$

Putting equation (1.7) into equation (1.1), and setting $T_{\mu\nu}=0$ in equations (1.1) and (1.6), we have

$$G_{\mu\nu} = -E_{\mu\nu}, \qquad (1.8)$$

and

$$D^{\nu}E_{\mu\nu} = 0. \qquad (1.9)$$

We can see that the tensor $E_{\mu\nu}$ acts as a "matter fluid" induced at the brane by the propagation of bulk gravitons. We can define an effective energy momentum tensor as

$$T_{\mu\nu}^{eff} = -\frac{1}{8\pi G_N}E_{\mu\nu}, \qquad (1.10)$$

so that the gravity field equations for the vacuum brane world looks like ordinary Einstein's equations in the presence of matter

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{eff}. \qquad (1.11)$$

III- INTER BRANE TRAVERSABLE WORMHOLE

To describe the spacetime manifold for a non rotating, spherically symmetric, horizon free wormhole with spatial $\mathbb{R}\times S^2$ topology we use the following line element:

$$ds^2 = -e^{2\phi(l)}dt + dl^2 + R(l)^2\left[d\theta^2 + sen^2\varphi d\varphi^2\right], \qquad (2.1)$$

where l represents proper radial distance, and $R(l)$ represents the radius of the wormhole's "tunnel" as a function of proper radial distance. For simplicity we assume the ultrastatic form of this metric by putting $\phi = 0$, then we have

$$ds^2 = -dt^2 + dl^2 + R(l)^2 d\Omega^2. \qquad (2.2)$$

The radius of the wormhole's throat is defined as the unique minimum value of the radial function profile $R(l)$

$$r_0 = \min\{R(l)\}. \qquad (2.3)$$

Without any loss of generality, we put $l=0$ at the point where $R(l)=r_o$.

The non zero components of the Einstein's tensor for the metric (2.2) are given by

$$G_{\hat{t}\hat{t}} = -\frac{2R''}{R} + \frac{1-(R')^2}{R^2}, \qquad (2.4)$$

$$G_{\hat{l}\hat{l}} = -\frac{1-(R')^2}{R^2}, \qquad (2.5)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \frac{R''}{R}, \qquad (2.6)$$

where the prime denotes derivative with respect to proper radial distance l.

For the weyl five dimensional tensor we choose a spherical symmetrical diagonal form with elements $E_{\mu\nu} = \left[E_{\hat{t}\hat{t}}, E_{\hat{l}\hat{l}}, E_{\hat{\theta}\hat{\theta}}, E_{\hat{\varphi}\hat{\varphi}}\right]$. This tensor has only two degrees of freedom because of the traceless condition and equation (1.9). We choose as the independent components $E_{\hat{t}\hat{t}}$ and $E_{\hat{l}\hat{l}}$.

Now we insert equations (2.4)-(2.6) and the tensor $E_{\mu\nu}$ in equation (1.8) and using (1.10), obtaining

$$\rho_{eff} = T_{\hat{t}\hat{t}}^{eff} = \frac{1}{8\pi G_N}\left[-\frac{2R''}{R} + \frac{1-(R')^2}{R}\right] \qquad (2.7)$$

$$P_{\hat{l}}^{eff} = T_{\hat{l}\hat{l}}^{eff} = \frac{1}{8\pi G_N}\left[-\frac{1-(R')^2}{R}\right] \qquad (2.8)$$

For the two independent components of the effective energy momentum tensor. We see immediately the following relation

$$\rho_{eff} + P_{\hat{l}}^{eff} = -\frac{1}{8\pi G_N}\left(\frac{R''}{R}\right) \qquad (2.9)$$

But we know that at the wormhole's throat $R(l) = r_0$ is a minimum value of $R(l)$, so we have

$$R''\Big|_{R=r_0} = \frac{d^2R}{dl^2}\Bigg|_{R=r_0} > 0, \qquad (2.10)$$

and

$$\left(\rho_{eff} + P_{\hat{l}}^{eff}\right)\Big|_{R=r_0} < 0, \qquad (2.11)$$

at the throat $R=r_o$.

In order to find explicit expressions for the components of the effective energy momentum tensor, generated by bulk weyl curvature, as functions of the radius of the throat of the wormhole, we assume a hyperbolic radial profile function, near the throat, as

$$R(l)=\frac{r_0}{K}\sqrt{l^2+K^2}\ ,\ K>0 \qquad (2.12)$$

where K is a parameter related to the curvature κ of the radial profile function and the wormhole's throat radius by the equation

$$K=\sqrt{\frac{r_0}{\kappa}}\ . \qquad (2.13)$$

We see from equation (2.13) that the higher the value of K the lower the spatial curvature in the proper radial direction. So in the limit $K\rightarrow\infty$ the radial profile function becomes constant, and equal to the radius of the throat

$$\lim_{K\rightarrow\infty} R(l)=r_0\ . \qquad (2.14)$$

In this limit we have a wormhole with cylindrical spatial geometry and no curvature in the proper radial direction. In order for the spatial geometry to tend to an appropriate asymptotically flat limit, we must impose another constraint given by the limit

$$\lim_{l\rightarrow\infty}\frac{R(l)}{l}=1\ . \qquad (2.15)$$

That is $r(l)=l+O(l)$ [8]. Condition (2.15) force us to set $K=r_0$, so our radial profile function is now given by

$$R(l)=\sqrt{l^2+r_0^2}\ . \qquad (2.16)$$

It can be seen that equation (2.16) satisfies the asymptotic condition (2.15) and reduces to the wormhole's throat radius at $l=0$.

Using equations (2.16), (2.4)-(2.6), (1.10) and (1.11), we obtain the components of the effective energy momentum tensor, induced by bulk weyl curvature on the brane,

$$\rho_{eff}=\frac{1}{8\pi G_N}\left[\frac{-l^4-3l^2r_0^2-2r_0^4+(l^2+r_0^2)^2}{(l^2+r_0^2)^3}\right], \qquad (2.17)$$

$$P_{\hat{l}}^{eff}=\frac{1}{8\pi G_N}\left[\frac{l^2}{(l^2+r_0^2)^2}-\frac{1}{(l^2+r_0^2)}\right], \qquad (2.18)$$

$$P_{\hat{\theta}}^{eff}=P_{\hat{\varphi}}^{eff}=\frac{1}{8\pi G_N}\left[\frac{r_0^2}{(l^2+r_0^2)^2}\right] \qquad (2.19)$$

By adding equations (2.17) and (2.18) we get, in terms of the principal pressures, the apparent violation of the null energy condition

$$\rho_{eff}+P_{\hat{l}}^{eff}=\frac{1}{8\pi G_N}\left[-\frac{r_0^2}{(l^2+r_0^2)^2}\right]<0 \qquad (2.20)$$

IV- CONCLUSION

In this paper we have constructed a model for an ultrastatic traversable wormhole with a hyperbolic radial function profile. A parameter K related to the spatial curvature of the radial profile function was introduced. Boundary conditions in our wormhole spacetime force us to set K equal to the radius of the throat of the wormhole. Finally the components of the effective energy momentum tensor were calculated explicitly as functions of the proper radial distance and the radius of the throat of the wormhole. The wormhole solution presented in this paper is a solution to the effective vacuum gravity equations so the apparent violation to the null energy condition is an effect of gravitons propagating in the bulk spacetime and not due to any exotic matter in the brane

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