The Structure of a Composite Galaxy Model

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Abstract – The algorithms and numerical methods describes the way of implementing a composite galaxy –bulge area, disk, and galaxy halo– which is modeled as a full three-dimensional entity, using spherical and disk models as primary support. The research models proposed were implemented in order to complement a GalaxSee program, parallel software for educational development of the portable cluster LittleFe has being implemented as well.

INTRODUCTION

In the past it has been different intents to simulate the cosmological expansion of the universe, every part of the process as one of many dynamic systems representations, but reality is that these tools and software demands an extremely high computational resources. Through the years the simulations ought to be better in terms of orders of magnitude, as high accuracy, robustness of the data collected including the improvement of analysis and data extraction from previous observations.

The most common tools for this type of simulations are the use of efficient mathematical models (Scientific Computing), and Supercomputers (Thousands of Parallel Computer with CPU's).

Given the high cost of commercial supercomputers, the monopoly of software (manufactured by the same vendor), and the increasing need for government / industry in hiring specialized personnel with more skills in the computer area; is producing an effect were the universities and the high performance computer research community to make a concerted effort to update curricula and manufacture their own parallel computers. Up to this day exists many high-performance computing (HPC) systems for educational purposes, like the purpose cluster system - which is available to serve the high demand trying to cover some of this computation needs [3], [5].

The Polytechnic University of Puerto Rico (UPPR) have a platform academically in use to teach courses of software and hardware, this platform is used under the parallelism techniques to supply services for the sub-graduate and graduate levels [1]. The laboratory facilities at the UPPR uses *Bootable Cluster CD* (BCCD) as a software platform, providing the necessary tools to teach HPC. This software can be run on a single system (portables PC) or on a network [4]. The cluster used is *LittleFe* compounded of 6 nodes, supporting shared memory parallelism (OpenMP), distributed memory parallelism (MPI), and allowing programmers to code al-gorithms for Graphics Processing Unit (GPU) on a Compute Unified Device Architecture (CUDA) and the Beowulf style portable cluster [3]. Both systems run with GNU / Linux-based operating system [2].

This paper describes an approximation for the construction of a galaxy composed of a disk / bulge area and halo, based on data collected from previous years of research (e.g. Verner 1983, Hernquist 1992, Dehnen 1993, Sotnikova 2004, and Barnes 2013). The density functions DF for each component of the galaxy are applied under the algorithm described [10] for the motion of the galaxy. The cluster *LittleFe* brought up by an educational packet software, including the use of the program *GalaxSee*; the program simulates the galaxies dynamics as well, this based a Newtonian model, and several numerical methods used to emulate the motion of particles. The program describes the basics for Galactic models, were the particles are randomly generated in different frames, spherical, cube and spiral among others.

This research was oriented to simulate the composite Galaxy model (disk, halo and bulge), the description of methodology, numerical methods and software tools used to obtain the visual representation of the Galaxy. The main idea is adding a more complex Galaxy model to the original program *GalaxSee*. The implementation has been done in MATLAB R2013b and the Wolfram Mathe-

matical used for simulation and as mathematical tools.

The organization of the paper is as follows: In MODEL resides the prescriptions used to construct the galaxy and in RESULTS is discussed the results of the numerical simulation. In CONCLUSIONS are summarized. The FUTURE WORK describes the future work and scientific approach of the simulation, lastly in the ACKNOWLEDGMENTS are the notes for appreciation and thank you to everyone who contributed in the research and project development.

MODEL

There are several ways to describe a galaxy model containing a stellar disk, bulge and surrounding dark matter halo. The components of the galaxy are represented in a cylindrical system.

The total gravitational potential can be written as:

$$\Phi_{total} = \Phi_{disk} + \Phi_{halo} + \Phi_{bulge} \quad (1)$$

The Galaxy model is formed by an exponential disk profile (Thin disk) [9] [12], for the bulge area the Hernquist model [11] was used, finally the halo model has been described by Dehnen [13]. In the following sections description of each components of our galaxy has been included.

Spherically Models

Assuming a spherical symmetry in space, in other words, where both potential $\Phi(r)$ and density $\rho(r)$ only depend on the distance to the center r, independent of the spherical angles θ and ϕ.

The total gravitational potential at radius r is generated by an arbitrary spherical symmetry density distribution, may be considered by the sum of the potential of spherical shells.

$$\Phi(r) = -\frac{G}{r}\int_0^r dM(t) - G\int_r^\infty \frac{M(t)}{dt} \quad (2)$$

Where $M(r)$ is the cumulative mass distribution, defined by the amount of mass that is included within the star cluster inclusive to a distance r from the star center.

$$M(r) = \int_0^r \rho(t)\,t^2 dt \quad (3)$$

The circular velocity is defined to be the speed of a test particle in a circular orbit at radius r.

$$v_c^2(r) = r\frac{d\Phi}{dr} = \frac{GM(r)}{r} \quad (4)$$

The particle's energy ε, it is the sum of kinetic energy $\varepsilon_{kin} = \frac{1}{2}mv^2$ and the potential energy $\varepsilon_{pot} = m\Phi(r)$. A particle at r can escape from the gravitational potential only if the kinetic (ε_{kin}) energy exceeds the value of the potential energy (ε_{pot}), then defined as an escape velocity:

$$\frac{1}{2}mv_{esc}^2 + m\Phi(r) = 0 \quad (5)$$

$$v_{esc}^2 = \sqrt{-2|\Phi(r)|}$$

For simple spherical models, augmented density is function of the potential only,

$$\rho(\Phi, r) \equiv \rho(r) \quad (6)$$

For such models, the distribution function (DFs) depend only on energy; then we say that the distribution function is *isotropic*. The approximation of DFs is calculated using the Eddington relation:

$$f(\varepsilon) = \frac{1}{2\sqrt{2}\pi^2}\frac{d}{d\varepsilon}\int_\varepsilon^0 \frac{d\Phi}{\sqrt{\Phi - \varepsilon}}\frac{d\rho}{d\Phi} \quad (7)$$

The radial velocity dispersions can be found from the density

$$\sigma_r^2 = \frac{1}{\rho(r)}\int_0^\Phi \rho(\Phi', r)\,d\Phi' \quad (8)$$

The research has been done using generic formulas [13] to cal-culate the density function, and the gravitational potential, where $\gamma=0,1$ for halo and bulge respectively.

$$\rho(r) = \frac{(1-\gamma)M}{4\pi}\frac{a}{r^\gamma(r+a)^{4-\gamma}} \quad (9)$$

$$\Phi(r) = \frac{GM}{a}$$

$$x\begin{cases} -\frac{1}{2-\gamma}\left[1 - \left(\frac{r}{r+a}\right)^{2-\gamma}\right] & \gamma \neq 2 \\ \ln\left(\frac{r}{r+a}\right) & \gamma = 2 \end{cases} \quad (10$$

The cumulative mass and circular velocity are:

$$M(r) = M\left(\frac{r}{r+a}\right)^{3-\gamma} \quad (11)$$

$$v_c^2(r) = \frac{GMr^{2-\gamma}}{(r+a_b)^{3-\gamma}} \quad (12)$$

The dispersion velocity is given by

$$\sigma_r^2 = GMr^\gamma(r+a)^{4-\gamma}\int_r^\infty \frac{r'^{1-2\gamma}}{(r'+a)^{7-2\gamma}}dr' \quad (13)$$

Bulge Model

The spherical Hernquist model [13] using the density function (9) for $\gamma=1$, calculated the density distribution profile for the bulge:

$$\rho(r) = \frac{M_b}{2\pi}\frac{a_b}{r(r+a_b)^3} \quad (14)$$

Where, M_b is the total mass of the bulge and a_b is the radial scale length. The spherical potential associate to this density profile is

$$\Phi_b(r) = -\frac{GM_b}{(r+a_b)} \quad (15)$$

The cumulative mass profile was obtained by integrating (3)

$$M(r) = \frac{M_b r^2}{(r+a_b)^2} \quad (16)$$

The circular velocity and escape velocity are defined by (4) and (5)

$$v_c^2(r) = \frac{GM_b r}{(r+a_b)^2} \quad (17)$$

$$v_{esc}^2 = \sqrt{\left|-2\left|-\frac{GM_b}{(r+a_b)}\right|\right|} \quad (18)$$

The density as function of the

potential is defined by

$$\rho(\Phi) = \frac{1}{2\pi a_b^3} \frac{\Phi^4}{(1-\Phi)} \quad (19)$$

Replacing equation (15) into Eddintong's formula equation (7) the distribution function (f_h) was calculated.

$$f_h(E) = \frac{1}{\sqrt{2}\pi^3 (GM_b a_b)^{\frac{3}{2}}}$$

$$x \frac{\sqrt{\varepsilon}(1-2\varepsilon)(8\varepsilon^2 - 8\varepsilon - 3)}{(1-\varepsilon)^2} + \frac{3\sin^{-1}(\sqrt{\varepsilon})}{(\varepsilon(1-\varepsilon))^{\frac{5}{2}}} \quad (20)$$

Where $E = \frac{v_b^2}{2} + \Phi$ and $\varepsilon = -\frac{a_b E}{GM_b}$ [13]

Calculating the dispersion velocity (13) value obtained as

$$\sigma_r^2 = -\frac{GM_b^2}{12 a_b}$$

$$x \left\{ \begin{array}{c} \frac{12r(1+a_b)^3}{a_b{}^4} \ln\left(\frac{r+a_b}{r}\right) \\ \left[12\left(\frac{r}{a_b}\right)^2 + 52\left(\frac{r}{a_b}\right) + 25\right] \end{array} \right\} \quad (21)$$

Halo Model

The spherical Dehnen models [13] under the density function (9) for $\gamma = 0$ calculated the density distribution profile for the h:

$$\rho(r) = \frac{3M_h}{2\pi} \frac{a_h}{(r+a_h)^4} \quad (22)$$

Where, M_h is the total mass of the halo and a_h is the radial scale length. The spherical potential associated to this density profile is

$$\Phi_h(r) = \frac{GM_h}{2a_h} \cdot \left[1 - \left(\frac{r}{r+a_h}\right)^2\right] (23)$$

The cumulative mass profile was obtained by integrating (3)

$$M(r) = \frac{M_h r^3}{(r+a_h)^3} \quad (24)$$

The circular velocity and escape velocity are defined by (4) and (5)

$$v_c^2(r) = \frac{GM_h r^2}{(r+a_h)^3} \quad (25$$

$$v_{esc}^2 = \sqrt{-2\left|\frac{GM_h}{2a_h} \cdot \left[1 - \left(\frac{r}{r+a}\right)^2\right]\right|} \quad (26)$$

The density function can be ex-

pressed in term of potential, if we define:

$$y = \frac{r}{(r+a)} \quad (27)$$

$$\Psi = \Phi \left(\frac{GM_h}{a}\right)^{-1} \quad (28)$$

Then calculating

$$\rho(\Phi) = \frac{3M_h}{2\pi a_h^3} (1-y)^4 \quad (29)$$

Where $y = y(\Psi) = [1 - 2\Psi]^{\frac{1}{2}}$

Replacing (23) into Eddintong's formula, (7) the distribution function (f_b) is obtained

$$f_b(E) = \frac{3M_h}{2\pi^3 (GM_h a_h)^{\frac{3}{2}}}$$

$$x \sqrt{2\varepsilon} \frac{(3-4\varepsilon)}{1-2\varepsilon} - 3\sin^{-1}\left(\sqrt{\frac{2\varepsilon}{1-2\varepsilon}}\right) (30)$$

Where $E = \frac{v_h^2}{2} + \Phi$ and $\varepsilon = -\frac{a_h E}{GM_h}$ [7]

Calculation of the dispersion velocity (13) produced under this constrains,

$$\sigma_r^2 = GM_h \frac{(a_h + 6r)}{30(a_h + r)^2} \quad (31)$$

Radial Distribution

The radial positions of particle i (r_i) is defined by $r_i = r_i \hat{r}_i$ where r_i is choose to solve the nonlinear (16), (24) by Newton's method and, $\hat{r}$ is the random number of uniform distribution on the unit sphere.

$$M(r_i) - x_i = 0 \quad (32)$$

The random number x_i is of Uniform distribution from the interval $[0,m(\infty))$, and $M(R)$ is the cumulative mass distribution defined in equation (3). The mass of particles is $m_i = m(\infty)$.

Velocities Distribution

The polar angle φ initializes a uniform distribution in the interval $[0,2\pi)$, and θ by the distribution $p(\theta) = \frac{1}{2}\sin(\theta)\sin(\theta)$ in the interval $[0,2\pi)$. Finally the module of the ve-

locity υ is randomly sampled by the Gaussian distribution.

$$f(v, \sigma_r) = \frac{1}{(2\pi)^{\frac{3}{2}} \sigma_r^3} v^2 exp\left(-\frac{v^2}{2\sigma_r^2}\right) (33)$$

The Rejection Sampling method is applied for rejection particles falling outside the maximum height of the function (f). When v_i is calculated, obtaining $v_i = v_i \hat{v}_i$ where $\hat{v}$ is the random number of uniform distribution on the unitary sphere.

Newton's Method

The numerical Newton's method is used to find the zeros of a nonlinear function $y = f(x)$. Given the an initial condition x_o, the nonlinear function is evaluated calculating $y_o = f(x_o)$ in a similarly the derivative of the function $y'_o = f'(x_o)$, the x intercept of tangent line, which is the new approximation to the zero, this is $x_1 = x_0 - \frac{y_0}{y_0'}$, the process continue until the change in the approximations:

$$\left|\frac{x_{k+1} - x_k}{x_{k+1}}\right| < \varepsilon \quad (34)$$

Where ε is the tolerance is sufficiently small.

We define the k-nth step of the method as:

$$x_{k+1} = x_k - \frac{y_k}{y_k'} \quad (35)$$

The method is applied initially matching the original equation to zero $y - f(x) = 0$. [8]

The Rejection Sampling Method

The method accepts random samples under a function, $f(X)$, where X is a random number from a Uniform Distribution between [0,1]. The maximum height (M) of the function (f) must first be calculated. Another random number is generated and a Uniform Distribution in [0,M]. [6]

$$\begin{cases} Accept\ the\ point\ (X,Y)\ if\ Y \le f(X) \\ or \\ Reject\ the\ point\ (X,Y)\ if\ Y > f(X) \end{cases} \quad (36)$$

Thin Disk Model

The particles that conforms the disk are represented under a cylindrical coordinate system, where the initial values R, z y ϕ are generated by specific distributions. The exponential disk profile is defined by

$$\rho_d(R,z) = \frac{M_d}{4\pi R_d^2 z_d} \exp\left(\frac{-R}{R_d}\right) sech^2\left(\frac{z}{z_d}\right) \quad (37)$$

Where, M_d is the disk mass, z_d is a vertical scale thickness and R_d is the radial scale length. The potential density profile will be approximated by [10]

$$\Phi_d(R,z) \approx \Phi_0(R) + \Phi_1(R,z)$$
$$= -\frac{GM_d}{R_d} y[I_0(y)k_1(y) - I_1(y)k_0(y)]$$
$$+ 4\pi G \rho(R,0) z_d^2 ln\left(\cosh\left(\frac{z}{z_d}\right)\right) \quad (38)$$

Where $y = \frac{R}{(2R_d)}$ and I_0, I_1, R_0, R_1 are Modified Bessel functions
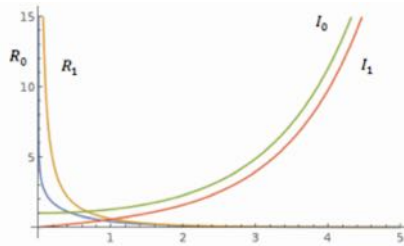
Figure 1
Modified Bessel Functions

Radial Distribution

The radial distribution of the particles are calculated from the exponential profile [8].

$$I = A_0\left[\exp\left(\frac{-R}{R_d}\right)\right] \quad (39)$$

Integrating the equation (18) obtains

$$B_{r_i} = \int_0^{R_i} A_0\left[\exp\left(\frac{-r}{R_d}\right)\right] dr$$
$$= A_0\left\{R_d - R_d \exp\left(\frac{-R_i}{R_d}\right)\right\} \quad (40)$$

Where

$$A_o = \left\{R_d - R_d \exp\left(\frac{-R_{max}}{R_d}\right)\right\}^{-1} \quad (41)$$

Where R_{max} is the maximun extension of the disk. Using the Newton's method to solve the following nonlinear function

$$f(R_i) = A_0\left\{R_d - R_d \exp\left(\frac{-R_i}{R_d}\right)\right\} - B_{r_i} \quad (42)$$

Where B_{r_i} is a random number [0,1).

Azimuthal Distribution

The angle ϕ_i assigned at each radial position R_i is defined by B_{ϕ_i}, where is calculated around of disk, with the equation [8].

$$\phi_i = B_{\phi_i} \times 2\pi \quad (43)$$

Where $B_{\phi_i} \in [0,1)$.

Height Distribution

Height distributions for brightness has a profile

$$I(z) = I_o sech^2\left(\frac{z}{z_d}\right) \quad (44)$$

Where I is the surface brightness, I_o is the central surface brightness, z is the height above the disk, and z_d is the scale height for the surface brightness. Integrating (44):

$$B_{z_i} = \int_{z_{min}}^{z_i} A_z sech^2\left(\frac{z}{z_d}\right) dz \quad (45)$$

Where A_z is constant and for the height of the disk $z_{min} = -z_{max}$, finally obtaining [8]

$$z_i = \frac{z_d}{2}\left[\frac{1+\tanh\left(\frac{z_{max}}{z_d}\right)(2B_{z_i}-1)}{1-\tanh\left(\frac{z_{max}}{z_d}\right)(2B_{z_i}-1)}\right] \quad (46)$$

Velocities Distribution

Transforming the equations of potential bulge and the halo in cylindrical coordinates, replacing $r = \sqrt{R^2 + z^2}$ to obtain:

$$\Phi_b(R,z) = -\frac{GM_b}{\left(\sqrt{R^2+z^2} + a_b\right)} \quad (47)$$

$$\cdot\left[1 - \left(\frac{\sqrt{R^2+z^2}}{\sqrt{R^2+z^2}+a_h}\right)^2\right] \quad (48)$$

We define the total potential as

$$\Phi_{tot} = \Phi_b + \Phi_h + \Phi_d \quad (49)$$

To compute the velocities of disk particles, is needed first to obtained $\sigma_z^2, \sigma_R^2, \sigma_\phi^2$ and $\langle v_\phi \rangle$, the radial and vertical components of the veloci-

ty of a particle R_i are obtained from Gaussian Distribution

$$f(v,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma}\right) \quad (50)$$

Where $\mu = 0$ and variance $\sigma = \sigma_z$ or $\sigma = \sigma_R$. The azimuthal component obtained by (43) with $\mu = \frac{\langle v_\phi \rangle}{R}$ and variance $\sigma = \sigma_\phi$.

The vertical velocity for disk dispersion σ_z^2

$$\sigma_z^2(z) = \frac{GM_d z_d}{2R_d^2}\exp\left(-\frac{R}{R_d}\right) \quad (51)$$

For disk radial dispersion velocity σ_z^2

$$\sigma_R^2(R_{ref}) = \left(Q_t \frac{3.36G \sum(R_{ref})}{\kappa(R_{ref})}\right) Q_t > 1 \quad (52)$$

Where Q_t is the Toomre parameter [10] at reference radius $R_{ref} \sim 2 - 3R_d$, $\sum(R)$ is the surface density profile of the disk

$$\sum(R) = \frac{M_d}{2\pi R_d^2}\exp\left(-\frac{R}{R_d}\right) \quad (53)$$

And $\kappa(R)$ is the epicyclic frequency defined as

$$\kappa(R) = \frac{\partial^2 \Phi_{tot}}{\partial R^2} + \frac{3}{R}\frac{\partial \Phi_{tot}}{\partial R} \quad (54)$$

The azimuthal velocity of disk dispersion σ_ϕ^2 by means of epicyclic approximation.

$$\sigma_\phi^2(R) = \sigma_R^2(R)\frac{\kappa^2(R)}{4\Omega^2(R)} \quad (55)$$

Where $\Omega^2(R)$ is the circular angular velocity

$$\Omega^2(R) = R^{-1}\frac{\partial \Phi_{tot}}{\partial R} \quad (56)$$

And, assuming an exponential distribution of the radial velocity [9], we defined the mean rotation velocity as

$$\langle v_\phi \rangle^2 = v_c^2 + \sigma_R^2\left(1 - \frac{\kappa^2(R)}{4\Omega^2(R)} - 2\frac{R}{R_d}\right) \quad (57)$$

Where v_c is the circular velocity

$$v_c^2(R) = R\frac{\partial \Phi_{tot}}{\partial R} \quad (58)$$

Once the velocities are calculated, the distribution function (f_d) of disk can be obtained

$$f_d\left(R,\phi,z,v_R,v_\phi,v_z\right) \propto \rho(R,\phi,z)\mathcal{H}\left(\frac{v_R}{\sigma_R(R)}\right)$$

$$\cdot\,\mathcal{H}\left(\frac{v_\phi-\langle v_\phi\rangle}{\sigma_\phi(R)}\right)\mathcal{g}\left(\frac{v_z}{\sigma_z(R)}\right)$$

Where v_R, v_ϕ and v_z are corresponding velocities in the radial, azimuthal and vertical direction. The $\langle v_\phi\rangle$ is the mean rotation velocity, while σ_R (R), $\sigma_\phi(R)$ and σ_z (R) are dispersions in the radial, azimuthal, and vertical directions, respectively. The functions $H(x)$ and $g(x)$ are defined by [13]

$$g(x) \propto exp\left(-\frac{1}{2}x^2\right) \quad (60)$$

$$\mathcal{H}(x) \propto exp\left(-\frac{1}{2}\left(\frac{x}{c}\right)^2 - \frac{1}{4}\left(\frac{x}{c}\right)^4\right) \quad (61)$$

Where c is fixed by requiring that

$$\int dx x^2\ \mathcal{H}(x) = \int dx \mathcal{H}(x) \quad (62)$$

RESULTS

The models were programmed with MATLAB R2013b and mathematical calculations using the Wolfram Mathematica. The numerical values for the experiments were calculated [13].

Experiment for Bulge

Generating the radial positions (r_i), with the values $M_b = 0.625$, $a_b = 0.02$ and the tolerance ($\epsilon = 10^{-8}$) applied with the Newton's method, the case for 10,000 particles is represented on figure 2 and 3.
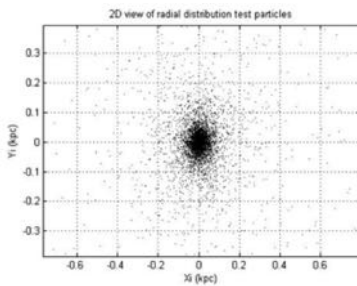

Figure 2
Face on Bulge

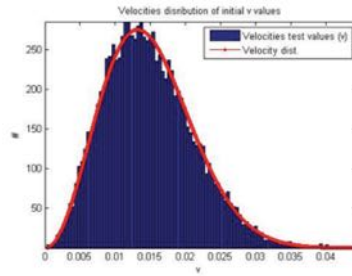The velocities of bulge are distributed through Gaussian distribution (12).

Figure 3
Velocities Initial of Bulge

Experiment for Thin Disk

Generating the initial values for R_i, ϕ_i and z_i the distributions were obtained, the test parameters values are R_d = 1.12 *kpc*, R_{max}=10 *kpc*, z_d = 0.00075 for a tolerance ($\epsilon = 10^{-8}$) applied by Newton's method, the case for 10,000 particles. Figure 4, 5, 6 and 7.
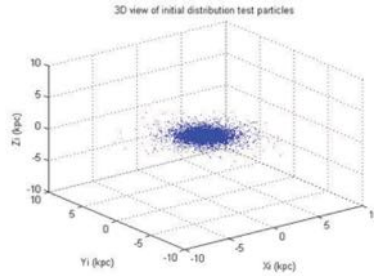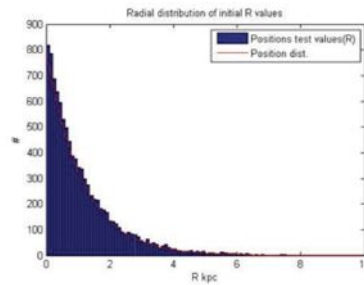

Figure 4
Disk Galaxy 3D Model

Figure 5
Radial Particles Distribution
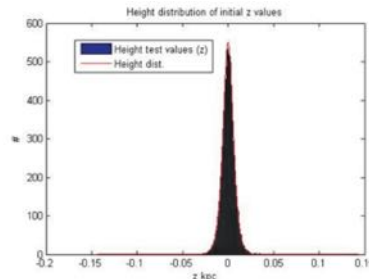

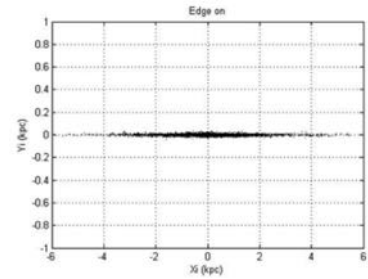**Figure 6**
Heights Particles Distributions

Figure 7
Disk Model

CONCLUSIONS

Models for the simulation of galaxies are particularly complex, the reason of this complexity is due to the number of parameters and mathematical equations needed, as well as the use of advance computational resources. The optimal use of time and resources has been done using algorithms that are parallelizable. The work presented is just an advance to this complex model, still the velocities of the disk and the distribution function of the disk were not calculated. The Halo calculation employed in this research are the same as those of the bulge area of the galaxy. The main purpose of this work is to continue, in order to finish the model and transport the data to run in the cluster *LittleFe*.

FUTURE WORK

The techniques presented describes the educational program *GalaxSee*, which are numerical methods solving the Newtonian system of particles motion in Cartesian 2D and 3D coordinates. For future work is to provide the solution of equation of particle motion using the algorithm with the distribution functions model, in cylindrical coordinate. Code this model in Unix platform to be done using in the cluster LitleFe. Also will be needed to include the equations for inclination of galaxies, all those calculations based on information

from astronomical databases, and finally do the research model merging the galaxies.

ACKNOWLEDGMENTS

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