Stirling Engine Mechanism Analysis as a Function of Internal Pressure Variation

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Abstract *— The Stirling heat engine was conceived in the in the early 1800's as an alternative to the steam engine. It is a regenerative close-cycle engine which utilizes the expansion and contraction of a working fluid at different temperatures, in such a way that there is net conversion of thermal energy to mechanical work. Noted for its high efficiency, robustness, and heat source versatility, but also for its low specific output, poor standardization, and a lack of understanding of its operation by the general public, a more exhaustive study of its performance and optimization is of great interest to the power generation and waste heat recovery industry [1]. The purpose of this project is to develop a design tool to calculate the engine parameters necessary to perform a kinematic assessment of the engine components, using the Schmidt analysis theory.*

Key Terms *— Design, Stirling Engine, Schmidt Analysis, Tool.*

INTRODUCTION

In recent times, the demand for more efficient methods of energy conversion has been increasing. The so-called "green technologies" and more efficient energy production methods are recently getting more attention than ever, in part due to increasing non-renewable energy costs and also due to the desire to decrease environmental pollution. On the other hand, any effort done to increase a country's energy production capacity will have a direct positive in its economic growth [2]. This is especially important in developing countries, where energy production is a key enabler of economic development.

Recovery of waste heat is another way to increase the efficiency of energy production systems, including power generation plants, process industries, and automotive applications. There are many methods that can be used to recover waste heat, including Stirling engines. Stirling engines are mechanical devices that work on the Stirling thermodynamic cycle, which is a closed regenerative cycle. The use of Stirling engines is desirable due to their high efficiency and their versatility, including the use of different working fluids, such as air, hydrogen, helium, and nitrogen, among others. Another advantage of the Stirling engine is its ability to operate with any form of thermal energy, making it an attractive alternative in both the renewable energy and heat recovery sectors.

Stirling engines are known for their efficient energy conversion, with efficiencies of more than 30% for optimized applications. However, in order to increase the efficiency and power to weight ratios, several design parameters need to be optimized, such as the operating pressure, piston geometry, regenerator design, and dead volume minimization.

The analysis methodology used throughout this project is based on the Schmidt or isothermal analysis theory and adapted for the alpha-type Stirling engine (Figure 1). The Schmidt analysis method is very useful during preliminary Stirling engine development, and is based on the isothermal expansion and compression of an ideal gas. The following assumptions must be observed when using the Schmidt analysis method [3]:

- There is no pressure loss in the heat-exchangers and there are no internal pressure differences.
- The expansion process and the compression process change isothermally.
- The working fluid behaves as an ideal gas.

- There is perfect regeneration.
- Expansion side dead volume temperature is equal to the expansion gas temperature, and compression side dead volume temperature is equal to the compression gas temperature.
- The regenerator temperature is an average of the expansion gas temperature and the compression gas temperature.
- The volume in the expansion space and in the compression space changes sinusoidally.

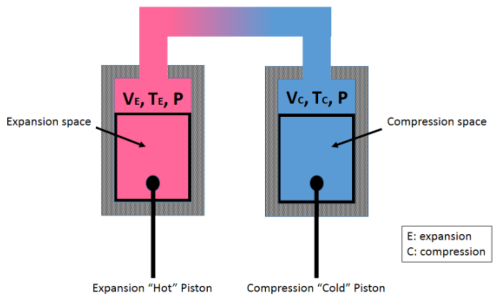

Figure 1
Alpha-Type Stirling Engine Diagram

The preliminary Stirling engine design tool developed on this project helps the engine designer to determine the theoretical engine performance and kinematic characteristics based on a set of pre-defined input parameters. It relates the engine pressure and temperature with cylinder geometry and crank-slider kinematics.

ENGINE DESIGN ASSUMPTIONS

The following assumptions are observed throughout the engine design methodology discussed here. They will be left as opportunities for future improvement of the tool:

1. The Stirling engine type covered under on this project is the alpha-type engine.
2. No regenerator component will be considered for this engine design.
3. The physical dimensions of the expansion (hot) cylinder and compression (cold) cylinders of the engine are assumed to be equal.
4. There is no offset between the piston axis and the crank axis (no *désaxé* effect, e=0).

PROGRAMMING PLATFORM

The purpose of this project is to determine the kinematic behavior of an alpha-type Stirling engine as a function of internal pressure variation. To achieve this purpose, a tool was developed to help visualize how these parameters interact in the preliminary design of a Stirling engine. This tool was created in Microsoft Excel due to the ubiquity of the software, it eases of use, and its expandability. It is expected and encouraged to continue to improve the tool in the future to make it more mature and to give more accurate engine performance predictions.

DESIGN METHODOLOGY

This Stirling preliminary design tool will combine the engine performance parameters calculated by the Schmidt analysis method with a mechanism analysis of the crank-slider-piston assembly. This will help the user determine the theoretical power and efficiency of the engine model, and the forces acting on relevant points in the mechanism, such as the force exerted by the piston on the slider and the torque acting on the crank. To achieve this, a set of equations will be defined so the user can vary the engine design parameters as needed and compare how each one of them affect engine performance.

ENGINE CYLINDER PARAMETERS & EQUATIONS

The Stirling preliminary design tool requires an initial set of physical piston dimensions (piston diameter *D*, crank radius *r*, connecting arm length *L, and* piston offset *e, in meters*) to determine the inner physical dimensions of the engine cylinders, i.e. the piston stroke and swept volume for both the hot and cold cylinders. These are defined as shown in the equations below [4]:

Piston stroke:

$$s = r\left[\sqrt{\left(\frac{l}{r}+1\right)^2 - \left(\frac{e}{r}\right)^2} - \sqrt{\left(\frac{l}{r}-1\right)^2 - \left(\frac{e}{r}\right)^2}\right] \quad (1)$$

Equation (1) reduces to simply $s = 2r$, when $e = 0$, (see Engine Design Assumption #4).

Piston swept volume:

$$V_{SE} = V_{SC} = \frac{\pi D^2}{4} * s \qquad (2)$$

REQUIRED INPUTS FOR SCHMIDT ANALYSIS

The Schmidt analysis method requires the following parameters to be supplied by the user to determine the engine performance, as defined in Table 1:

Table 1
Required Inputs for Schmidt Analysis

Name	Variable
Dead space of expansion space (m^3)	V_{DE}
Dead space of compression space (m^3)	V_{DC}
Cylinder phase angle (radians)	α
Mean Pressure (Pa)	P_{mean}
Expansion Gas Temperature (K)	T_E
Compression Gas Temperature (K)	T_C
Engine Speed (RPM)	N

The parameters above will be introduced in the following sets of equations to determine the instantaneous total gas volume and pressure, net work, power, and overall engine efficiency. The preliminary design tool will solve these equations as a function of the crank angle (φ) to create the corresponding P-V diagram and plot the total engine pressure, total gas volume, piston force and crank torque as a function of the crank angle.

Instantaneous expansion volume [4]:

$$V_E(\varphi) = \left(\frac{\pi D^2 r}{4}\right) * \sqrt{\left(\frac{l}{r}+1\right)^2 - \left(\frac{e}{r}\right)^2} - \left\{\cos(\varphi) + \sqrt{\left(\frac{l}{r}\right)^2 - \left[\sin(\varphi) - \frac{e}{r}\right]^2}\right\} \qquad (3)$$

Instantaneous compression volume:

$$V_C(\varphi) = \left(\frac{\pi D^2 r}{4}\right) * \sqrt{\left(\frac{l}{r}+1\right)^2 - \left(\frac{e}{r}\right)^2} - \left\{\cos(\varphi - \alpha) + \sqrt{\left(\frac{l}{r}\right)^2 - \left[\sin(\varphi - \alpha) - \frac{e}{r}\right]^2}\right\} \qquad (4)$$

Total instantaneous gas volume:

$$V_T(\varphi) = V_E(\varphi) + V_{DE} + V_C(\varphi) + V_{CE} \qquad (5)$$

Temperature ratio [3]:

$$t = \frac{T_C}{T_E} \qquad (6)$$

Equations (7) to (9) define the ratios between the expansion and compression swept volumes, and the ratios between the dead volumes and the swept volumes for the expansion and compression volumes:

Swept volume ratio:

$$v = \frac{V_{SC}}{V_{SE}} \qquad (7)$$

Dead expansion volume ratio:

$$X_{DE} = \frac{V_{DE}}{V_{SE}} \qquad (8)$$

Dead compression volume ratio:

$$X_{DC} = \frac{V_{DC}}{V_{SC}} \qquad (9)$$

Now using the following definitions:

$$a = tan^{-1}\left(\frac{v \sin(\alpha)}{t \cos(\alpha)}\right) \qquad (10)$$

$$S = t + 2tX_{DE} + v + 2X_{DC} \qquad (11)$$

$$B = \sqrt{t^2 + 2tv * \cos(\alpha) + v^2} \qquad (12)$$

$$c = \frac{B}{S} \qquad (13)$$

The *instantaneous gas pressure* can then be calculated as a function of crank angle using the mean pressure P_{mean}:

$$P(\varphi) = \frac{P_{mean}\sqrt{S^2 - B^2}}{S - B\cos(\varphi - \alpha)} = \frac{P_{mean}\sqrt{1 - c^2}}{1 - c\cos(\varphi - \alpha)} \qquad (14)$$

The *work* (in joules) (area under the curve of the P-V diagram, see Figure 2) in the expansion and compression spaces can be calculated based on the mean pressure P_{mean}:

$$W_E = \frac{P_{mean} V_{SE} \pi c \sin(\alpha)}{1+\sqrt{1-c^2}} \tag{15}$$

$$W_C = -\frac{P_{mean} V_{SE} \pi c t \sin(\alpha)}{1+\sqrt{1-c^2}} \tag{16}$$

The *net work* per cycle is then

$$W_i = W_e + W_c \tag{17}$$

The *expansion power* L_E, *compression power* L_C, *and net engine power* L_i (in watts) is similarly defined by

$$L_E = W_E \frac{N}{60} \tag{18}$$

$$L_C = W_C \frac{N}{60} \tag{19}$$

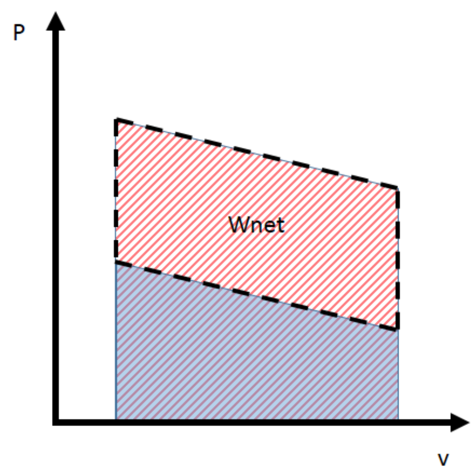

Figure 2
Simplified Stirling Engine PV Diagram

In the Figure 2, the Net Work is the difference between the areas below the expansion work (red lines) and the compression work (blue region).

$$L_i = W_i \frac{N}{60} \tag{20}$$

The *engine efficiency* is determined by

$$\eta = \frac{W_i}{W_E} = 1 - t \tag{21}$$

KINEMATIC ANALYSIS [5]

The crank slider mechanism analysis (Figure 3), used in the majority of internal combustion engines, can be used for the kinematic analysis of our Stirling engine [5].

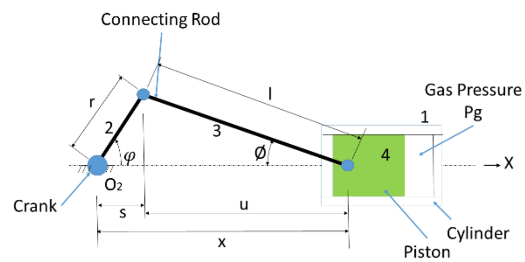

Figure 3
Crank-Slider Mechanism.

The *gas force* acting on top of the piston surface due to the gas pressure is defined as:

$$F_g(\varphi) = \frac{\pi}{4} P(\varphi) D^2 \tag{22}$$

Using the trigonometry definition in (23), the forces acting on the different components of the crank-slider mechanism can be obtained (see Figure 4):

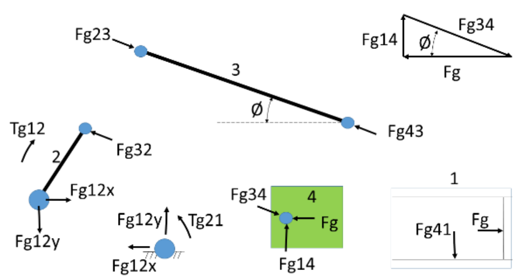

Figure 4
Crank-Slider Mechanism Free Body Diagrams

$$\tan(\emptyset) = \frac{r \sin(\varphi)}{l\sqrt{1-\left[\frac{r}{l}\sin(\varphi)\right]^2}} \tag{23}$$

$$F_{g14} = F_g \tan(\emptyset)\, \hat{j} \tag{24}$$

$$F_{g34} = F_g \hat{\imath} - F_g \tan(\emptyset)\, \hat{j} \tag{25}$$

$$F_{g41} = -F_{g14} \tag{26}$$

$$F_{g43} = -F_{g34} \tag{27}$$

$$F_{g23} = -F_{g32} \quad (28)$$

$$F_{g32} = -F_{g23} \quad (29)$$

$$F_{g32} = -F_{g34} = F_g\,\hat{\imath} - F_g \tan(\emptyset)\,\hat{\jmath} \quad (30)$$

The *driving torque* [6] at the crank due to the gas force can be closely approximated using the following equation:

$$T_g(\varphi) \cong F_g(\varphi) r \sin(\varphi) \left[1 + \frac{r}{l}\cos(\varphi)\right] \quad (31)$$

This torque is independent of the *inertial torque*, which is dependent of the translating mass of the mechanism components and negligible at low engine speeds. The calculation of this torque is left as future work.

TOOL USAGE

The tool is divided into 3 parts contained in 3 different tabs in Excel:

- **Inputs-Outputs**: On the top portion of this tab the user enters the required geometry parameters of the engine, including cylinder and connector rod geometry. Other engine parameters required for the Schmidt analysis, such as engine mean pressure, hot and cold temperatures, cylinder phase angle, and engine speed, are introduced here. On the bottom portion of this tab the calculated engine parameters are shown, including net work, power, and overall efficiency.
- **Engine Charts**: This tab includes the engine PV diagram and engine parameters as a function of crank angle. This tab also includes the forces acting on the engine components as part of the kinematic analysis. The charts on this tab help the user visualize the engine behavior and gain a better understanding of the pressure and volume variations on each phase of the engine cycle.
- **Calculations**: User input is not required on this tab, as it contains the back-end calculations shown in the “Inputs-Outputs” and “Engine Charts” tabs. However, continuous improvement is expected on this tool, so the Schmidt analysis equations on this tab are subject to change eventually.

Data input in the “Inputs-Outputs” tab is straight forward. All cells are well labeled and color coded.

- Green cells are required user input.
- Greyed-out cells are place holders for future tool improvement. The “Notes” section provides more details.
- Grey cells are automatically calculated, no user modification is required.
- Blue cells are the engine performance results from the Schmidt analysis. The user should compare how different inputs affect these results, to determine the engine performance main “drivers”.

EXAMPLE CALCULATION

To provide an example of the results that can be obtained using the Schmidt analysis for an alpha-type Stirling engine, the following parameters were input into the Stirling preliminary design tool. The cylinder geometry is first calculated using equations (1) and (2) before starting the Schmidt theory calculations. Remember that the hot and cold cylinder stroke is the same, based in Engine Design Assumption #3.

Table 2
Inputs for Stirling Engine Analysis

Variable	Value
r	10 cm
D	5 cm
l	30 cm
V_{DE}	0.2 cm^3
V_{DC}	0.2 cm^3
α	90°
P_{mean}	100 kPa
T_E	200 °C = 473.15 K
T_C	25 °C = 298.15 K
N	70 RPM

Table 3
Calculated Cylinder Dimensions

Calculated Parameters – Cylinder Dimensions	
s_e (m)	0.2000
s_c (m)	0.2000
V_{se} (m^3)	0.0004
V_{sc} (m^3)	0.0004

The next set of calculated engine parameters are automatically solved using equations (6) to (13) in the first part of the "Calculated Parameters – Engine" section:

Table 4
Calculated Engine Parameters (I)

Calculated Parameters – Engine (I)	
t	0.630138434
v	1
X_{de}	0.000509296
X_{dc}	0.000509296
a	1.008510488
S	1.631798879
B	1.181979038
c	0.724341126

The second part of the "Calculated Parameters – Engine" section gives the engine work, power, and efficiency for the inputs provided in Table 2:

Table 5
Calculated Engine Parameters (II)

Calculated Parameters – Engine (II)	
W_e (J)	44.75070734
W_c (J)	-28.19914064
W_i (J)	16.5515667
L_e (W)	52.20915857
L_c (W)	-32.89899741
Power per cycle (W)	19.31016115
Efficiency	0.369861566

The "Engine Charts" tab provides a useful visualization aid to better understand how the engine pressure and volume vary as a function of the crank angle. The engine pressure vs. crank angle is plotted in Figure 5:

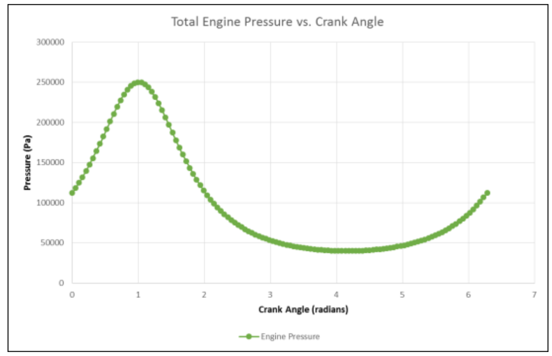

Figure 5
Engine Pressure as a Function of Crank Angle

The sum of the expansion and compression volumes and fixed dead volumes gives the total working gas volume in the engine. The cylinder phase angle determines the peak volume location in each cylinder along the engine cycle (Figure 6):

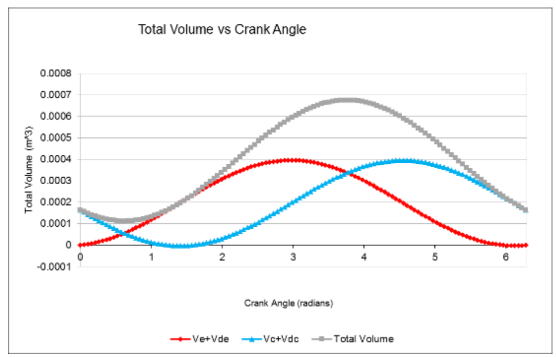

Figure 6
Working Gas Volume as a Function of Crank Angle.

The engine PV diagram is plotted in Figure 7 using the pressure and volume data from Figures 5 and 6. The area enclosed under the PV curve is the net work produced by the engine [6]:

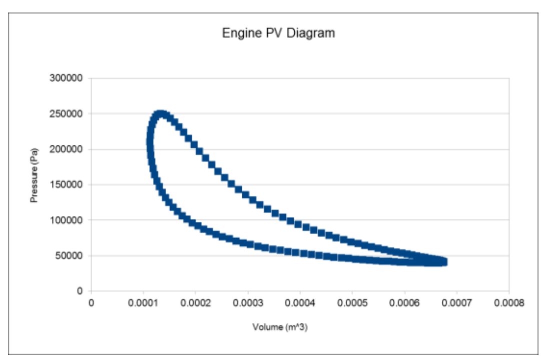

Figure 7
Stirling Engine PV Diagram

To complete the kinematic analysis of the engine, the gas force F_g and driving torque at the crank T_g are also plotted as a function of the crank angle (Figures 8 and 9) to determine the maximum force and torque acting on the piston and engine components defined in Figure 3. Remember that this driving torque is independent of the *inertial torque*, which is dependent of the translating mass of the mechanism components. The free body diagrams for these components are defined in Figure 4. For this example, the maximum gas force acting on the piston (either the hot piston or the cold piston, depending on the crank angle and the phase angle) is 491 N, and the maximum driving torque on the crank is 50 N•m. Also, the loads acting on each component are plotted on Figure 10, based on the results obtained from (24) to (30). A successful engine designer should be able to develop the engine design in such a way that the components can sustain these loads, and to apply a margin of safety on top of these preliminary results.

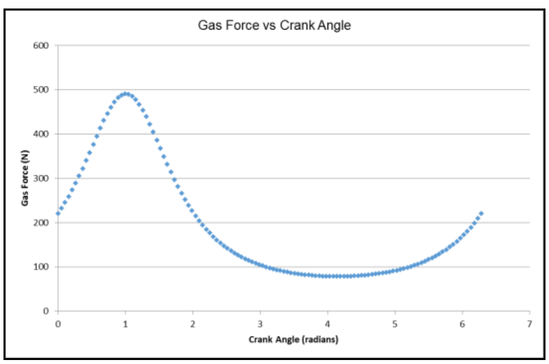

Figure 8

Gas Force Acting on Piston as a Function of Crank Angle.

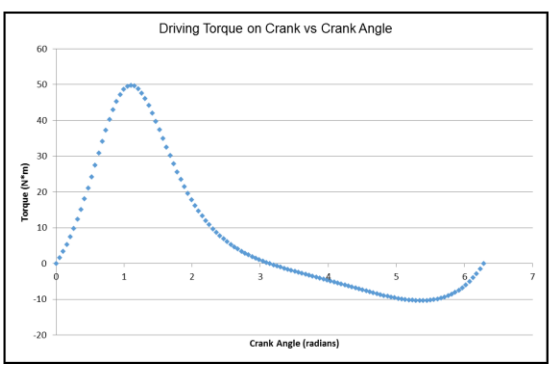

Figure 9

Gas Driving Torque on Crank as a Function of Crank Angle

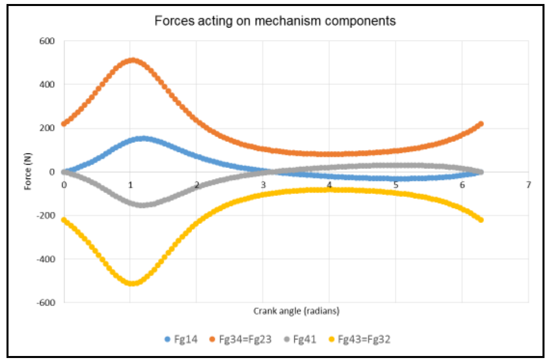

Figure 10

Forces Acting on Each Mechanism Component as a Function of Crank Angle.

FUTURE WORK

This preliminary design tool described in this project was created with the intent of helping to determine the main design parameters of a Stirling engine using the Schmidt theory, and to perform a kinematic analysis on the structural components. This tool by no means is do-it-all application, but a foundation to build upon. The following ideas are worth considering in order to improve the capability of the tool and the accuracy of its results:

- Although many of the equations used for this analysis are applicable to Stirling engines in general, an assessment and possible modifications of the concepts used on this project should be made to make the tool applicable to other Stirling engine types.
- The engine geometry for this project was simplified; the hot and cold cylinders are identical. It could be useful to optimize the tool for asymmetrical configurations. The "Inputs-Outputs" tab of the tool already provides a space for different connector rod and cylinder diameter values, but it left as an opportunity for improvement the modifications of the equations to use them.
- In many engine configurations there is an offset between the cylinder axis and the crank axis; this is called the *désaxé* effect. The tool initially included this value as an input, but due to increased complexity it was left out for the initial release. It should be added back in a

future revision.

- For this initial release, the regenerator component commonly found in Stirling engines was not included in the engine model studied. It should be easy to add the regenerator parameters to the existing equations and increase the usefulness of the tool.
- The thermodynamic aspect of the engine analysis was only briefly addressed by the Schmidt analysis by using an ideal gas. Also, the heat transfer phenomena occurring between the heat source and the hot cylinder, and between the cold cylinder and the surroundings are out of the scope of this task. It could be a beneficial to add the necessary equations and tabs to provide a more accurate representation of the transient thermal characteristics of the engine.
- The inertial torque can be calculated easily by calculating the mechanism component masses in pure translation. The textbook in [5] and [6] provides a good reference for this calculation.

CONCLUSION

The Schmidt Theory for the preliminary design analysis of Stirling engines is a powerful tool to determine their theoretical performance characteristics given an initial mean pressure and temperature difference, and a set of mechanism design inputs. For this project, an alpha-type Stirling engine was considered. The internal engine pressure variation is assumed to behave in a sinusoidal pattern, so it was derived as a function of the engine crank angle. This helps us determine all desired engine parameters as a function of a single independent variable, which simplifies the analysis. Likewise, the piston gas force and driving torque are obtained as a function of the engine pressure, which lets calculate the remaining kinematic forces acting of the engine, modeled as a crank slider mechanism.

Although the Schmidt Theory assumes ideal gas properties and isothermal gas expansion and compression, it is still very useful during preliminary engine development. Most importantly, the tool developed in this task helps to advance the general understanding of the Stirling engines in general, which can have a large impact in our society due to their advantages in efficient power generation capabilities. Efficient and cheap energy generation is one of the main drivers of economic growth.

Waste-heat recovery is another field of opportunity for Stirling engines due to their versatility in terms of the variety of heat sources that can be utilized for energy conversion, and their simplicity and low maintenance, which facilitates their use in remote areas.

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