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The generalized cross-product of vectors

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Abstract

The meaning of the scalar product of two vectors can be easily extended to a space of n dimensions. Similar generalization, related to the cross-product of vectors with more than three components, has not been formulated in matrix notation. This paper explicitly defines the result of the cross-product of vectors with an arbitrary number of components. The operation is based on the concept of the annulment matrix of a vector.

Sinopsis

El significado del producto escalar de dos vectores ha sido fácilmente extendido a un espacio de n dimensiones. Una generalización similar, pero relacionada con el producto vectorial de dos vectores, no se ha formulado usando notación matricial. Este trabajo define explícitamente la operación del producto vectorial de dos vectores con un número arbitrario de componentes. Tal definición se establece a partir del concepto de la matriz anulante de un vector.

Introduction

It has been stated that all concepts of three-dimensional geometry generalize to a vector of any finite order n [1]. However, concerning the cross-product of vectors such a generalization has not been defined and this type of operation does not find a simple counterpart in matrix algebra[2].

This paper presents the formulation of a matrix expression that defines the cross-product of two vectors of arbitrary order n. Obviously, such a formulation must coincide with the one currently used for the particular case in which n=3.

Furthermore, the generalization is extended to the definition of the cross-product involving (n-1) vectors in a space of n dimensions, in order to solve the following problem: given (n-1) vectors with n components each, $\mathbf{V}_1$, $\mathbf{V}_2$, ... $\mathbf{V}_{n-1}$, find a nonzero vector $\mathbf{V}_n$ that is normal to each of them.

The annulment matrix of a vector

Consider a vector **U** with n components. The annulment matrix of this vector will be represented by $[\mathbf{N}]_u$ and defined as follows:

$$[\mathbf{N}]_u = [\mathbf{I}] - \mathbf{U} \times \mathbf{U} / \mathbf{U}.\mathbf{U} \qquad (1)$$

where [I] stands for the identity matrix of order n and **U** x **U** indicates a matrix resulting from the tensor product of U by itself. The scalar product of U by itself, represented by **U.U**, leads to the square of its length $|\mathbf{U}|$.

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1 Bathe, K.J. and Wilson, E.L., 1976, *Numerical Methods in Finite Element Analysis*, Englewood Cliffs, New Jersey: Prentice-Hall, Inc,.

2 Zienkiewicz, O.C., 1977, *The Finite Element Method*, Third edition. London, Great Britain: Mc Graw-Hill Book Co. (UK) Ltd.

Accordingly, if u_k stands for the k-th component of **U**, then:

$$\mathbf{U}.\mathbf{U} = u_1^2 + ... + u_n^2 = |\mathbf{U}|^2 \qquad (2)$$

The appellation used for $[\mathbf{N}]_u$ comes from the fact that $[\mathbf{N}]_u U=\{\mathbf{0}\}$; that is, premultiplication of a vector by its annulment matrix yields a null vector. It is easy to verify that the i-th diagonal component of $[\mathbf{N}]_u$ is equal to $(|\mathbf{U}|^2-u_i^2)/|\mathbf{U}|^2$, while an off-diagonal component, in row i and column j, has the value $-u_iu_j/|\mathbf{U}|^2$.

Projections of a vector

Consider two vectors, **U** and **V**, with n components each one. The projection of any vector on the direction defined by another is related to the scalar product of both vectors and we may write:

$$\mathbf{V} \text{ component along } \mathbf{U} = (\mathbf{U}.\mathbf{V}/\mathbf{U}.\mathbf{U})\,\mathbf{U} \qquad (3)$$

When the annulment matrix of **U** premultiplies **V** the result is the projection of the latter on a direction normal to the former, that is:

$$\mathbf{V} \text{ component normal to } \mathbf{U} = [\mathbf{N}]_u\mathbf{V}=\mathbf{V}-(\mathbf{U} \times \mathbf{U}/\mathbf{U}.\mathbf{U})\,\mathbf{V} \qquad (4)$$

Accordingly, the scalar product involving **U** and $[\mathbf{N}]_u\mathbf{V}$ is identical to zero and the annulment matrix of **U** represents a linear transformation that maps **V** to the component of **V** normal to **U**.

It can be shown that the i-th component of $[\mathbf{N}]_u\mathbf{V}$ is equal to the expression $\{v_i - (\mathbf{U}.\mathbf{V}/\mathbf{U}.\mathbf{U})u_i\}$, in which u_i and v_i stand for the i-th component of **U** and **V**, respectively. Therefore, this vector coincides with one obtained by means of the Gram-Schmidt orthogonalization applied to the original vectors **U** and **V** [3].

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[3] Goodbody, A.M., 1982, *Cartesian Tensors*, New York, New York: John Wiley and Sons.

If n=3 it is evident that the length of **U** times the length of $[\mathbf{N}]_u\mathbf{V}$ yields the length of the vector that results from the cross product involving **U** and **V** since $[\mathbf{N}]_u\mathbf{V}$ expresses the projection of **V** on a direction normal to **U**. However, $[\mathbf{N}]_u\mathbf{V}$ is normal to **U** but not to **V** and therefore it can not be used to identify the direction of the "cross product" **U**x**V**. Nevertheless, this direction can be defined following the rules discussed below.

The cross-product of two vectors in n dimensions

In relation with vectors **U** and **V** previously considered, let $[\mathbf{N}]_{uv}$ represent the annulment matrix of vector $[\mathbf{N}]_u\mathbf{V}$. If $\{\mathbf{1}\}$ stands for a vector whose n components are equal to one, it may be shown that the vector resulting from the operation $[\mathbf{N}]_{uv}$ times $\{\mathbf{1}\}$ is normal to both original vectors, **U** and **V**. Accordingly, the three vectors **U**, $[\mathbf{N}]_u\mathbf{V}$ and $[\mathbf{N}]_{uv}[\mathbf{N}]_u\{1\}$ are pairwise normal and the direction parameters of the latter can be used to identify the direction of vector **W** resulting from the cross product **U**x**V**. As stated before, the length of **W** is obtained by multiplying the length of **U** by the length of $[\mathbf{N}]_u\mathbf{V}$.

Following above considerations an explicit matrix expression can be derived for the calculation of **W**. Let us first construct an (n x 3) matrix **[M]** with vectors $\{\mathbf{1}\}$, **U** and **V** as illustrated below:

$$[\mathbf{M}] = [\{\mathbf{1}\}\ \mathbf{U}\ \mathbf{V}] \tag{5}$$

and let us now define a (3 x 3) matrix [S] as the result of the product of **[M]** transpose times **[M]**, that is:

$$[\mathbf{S}] = [\mathbf{M}]^T\ [\mathbf{M}] \tag{6}$$

It is evident that [S] is symmetric and that each of its components is identified by a scalar product as follows:

$$\mathbf{S}_{11} = \{\mathbf{1}\}.\{\mathbf{1}\} = n \tag{7}$$

$$\mathbf{S}_{12} = \{\mathbf{1}\}.\mathbf{U} = u_1+u_2+ \ldots +u_n \quad (8)$$
$$\mathbf{S}_{13} = \{\mathbf{1}\}.\mathbf{V} = v_1+v_2+ \ldots +v_n \quad (9)$$
$$\mathbf{S}_{22} = \mathbf{U}.\mathbf{U} = u_1^2+u_2^2+ \ldots +u_n^2 \quad (10)$$
$$\mathbf{S}_{23} = \mathbf{U}.\mathbf{V} = u_1v_1+u_2v_2+ \ldots +u_nv_n \quad (11)$$
$$\mathbf{S}_{33} = \mathbf{V}.\mathbf{V} = v_1^2+v_2^2+ \ldots +v_n^2 \quad (12)$$

Scalar product expressions detailed above assume that the components of all vectors are written in reference to an orthogonal system of coordinates, that is, one in which the metric tensor **[G]** is equal to the identity matrix **[I]**. Had these vectors been referred to a non-orthogonal system of coordinates, the corresponding metric tensor **[G]** would have been taken into consideration in the calculation of each scalar product.

Finally, if **C** stands for a 3-component vector listing the first row cofactors of determinant $|\mathbf{S}|$, and D represents the numerical value of the same determinant, it may be shown that the following matrix expression gives the cross-product of two vectors **U** and **V** of order n:

$$\mathbf{U}\mathrm{x}\mathbf{V} = [\mathbf{M}]\mathbf{C}/\sqrt{D} \quad (13)$$

Moreover, if **W** denotes the resultant vector, the square of its length is equal to the first cofactor C_1, that is, $|\mathbf{W}|^2 = C_1$. If we are only interested in the direction of **W**, the denominator $\sqrt{D}$ can be omitted and the calculation can be limited to **[M]C**.

Let us examine whether the cross-product definition established above leads to the one currently used in the 3-dimensional space, that is, the particular case in which n=3. For this purpose, we define the three determinants shown in equation 14. It can be shown that in this case, $\sqrt{D} = d_1 + d_2 + d_3$ and the j-th component of **[M]C** is equal to d_j / $(d_1 + d_2 + d_3)$. Therefore, $W_j = d_j$, as prescribed by the cross-product rule in three dimensions.

$$d_1 = \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}, \quad d_2 = \begin{bmatrix} u_3 & u_1 \\ v_3 & u_1 \end{bmatrix}, \quad d_3 = \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \tag{14}$$

Numerical example involving 2 vectors in 4 dimensions

Let **U** and **V** transpose equal to [4 - 5 7 3] and [-11 9 3 2], respectively. Forming matrices [M] and [S] previously described:

$$[M] = [\ \{1\}\ U\ V\ ] = \begin{bmatrix} 1 & 4 & -11 \\ 1 & -5 & 9 \\ 1 & 7 & 3 \\ 1 & 3 & 2 \end{bmatrix} \tag{15}$$

$$[S] = [M]^T\ [M] = \begin{bmatrix} 4 & 9 & 3 \\ 9 & 99 & -62 \\ 3 & -62 & 215 \end{bmatrix} \tag{16}$$

The first row cofactors of $|S|$ are:

$$C_1 = 99x215 - 62x62 = 17441 \tag{17}$$
$$C_2 = -9x215 - 3x62 = -2121 \tag{18}$$
$$C_3 = -9x62 - 3x99 = -855 \tag{19}$$

Accordingly, **C** transposed is equal to [17441 - 2121 - 855] and the expansion of det **S** by the first row yields:

$$D = 4 \text{ x } 17441 - 9 \text{ x } 2121 - 3 \text{ x } 855 = 48110 \tag{20}$$

Performance of the operation $[M]C/\sqrt{D}$ leads to vector W equal to the cross-product involving vectors U and V:

$$\mathbf{W} = \mathbf{U} * \mathbf{V} = \frac{[\mathbf{M}]\ \mathbf{C}}{\sqrt{D}} = \frac{1}{\sqrt{48110}} \begin{bmatrix} 18362 \\ 20351 \\ 29 \\ 9368 \end{bmatrix} \tag{21}$$

The square of the length of **W** is given by the first cofactor of $|\mathbf{S}|$, that is:

$$|\mathbf{W}|^2 = C_1 = 17441 \tag{22}$$

In 3-dimensional space the following relationship exists between the lengths of 2 vectors **U** and **V**, the length of the vector obtained by the cross-product **U** x **V** and the result of the scalar product **U**.**V**:

$$(\mathbf{U}.\mathbf{V})^2 + |\mathbf{U} \times \mathbf{V}|^2 = |\mathbf{U}|^2 |\mathbf{V}|^2 \tag{23}$$

The generalized cross-product definition previously described preserves such relationship. In the numerical example under consideration we have that $(-62)^2 + 17441 = 99 \times 215$.

The cross-product of (n-1) vectors in n dimensions

The criteria described above can be extended to formulate a more proper generalization of cross-product definition involving (n-1) vectors, $\mathbf{V}_1$, $\mathbf{V}_2$, ..., $\mathbf{V}_{n-1}$, in a space of n dimensions. Such a formulation will establish the rule to obtain a vector $\mathbf{V}_n$ that is normal to the hyperplane spanned by the (n-1) given vectors.

First we shall construct an (n x n) matrix **[M]** using vectors {1}, $\mathbf{V}_1$, $\mathbf{V}_2$, ..., $\mathbf{V}_{n-1}$ to define its n columns as follows:

$$[\mathbf{M}] = [\{1\}\ \mathbf{V}_1\ \mathbf{V}_2 \cdots \mathbf{V}_{n-1}] \tag{24}$$

Then we shall calculate an (n x n) matrix [S] in such a way that its component in row i and column j is equal to the scalar product involving the i-th row of $[\mathbf{M}]^T$ and the j-th column of $[\mathbf{M}]$, that is:

$$[S] = [\mathbf{M}]^T[\mathbf{M}] \tag{25}$$

Finally, if **C** stands for an n-component vector listing the first row cofactors of determinant $|\mathbf{S}|$ and D is equal to the numerical value of the same determinant, the following matrix expression defines a new vector $\mathbf{V}_n$ which represents the cross-product related to the (n - 1) given vectors:

$$\mathbf{V}_n = [\mathbf{M}]\ \mathbf{C} / \sqrt{D} \tag{26}$$

It may be verified that the square of the length of $\mathbf{V}_n$ is equal to the first cofactor of $|\mathbf{S}|$, that is,

$$|\mathbf{V}_n|^2 = C_1 \quad . \tag{27}$$

A new symbol should be used when referring to this type of operation in order to avoid confusion with a multiple cross-product involving (n - 1) vectors in a 3-dimensional space. The author proposes to use an asterisk (*) rather than a cross (x) and the operation could be termed asterisk-product rather than cross-product. We would then summarize the contents of this work with the definition of a new type of product, the asterisk-product, as follows:

$$\mathbf{V}_n = \mathbf{V}_1 * \mathbf{V}_2 * \ldots * \mathbf{V}_{n-1} = [\mathbf{M}]\mathbf{C} / \sqrt{\det S} \tag{28}$$

where $[\mathbf{M}] = [\{1\}\ \mathbf{V}_1\ \mathbf{V}_2 \ldots \mathbf{V}_{n-1}]$, $|\mathbf{S}|$ is the determinant related to matrix $[\mathbf{M}]^T[\mathbf{M}]$ and **C** is a vector whose n components are equal to the first row cofactors of $|\mathbf{S}|$. Normalized expression of $\mathbf{V}_n$ can be obtained by dividing each component by $\sqrt{C_1}$.

Applications

The asterisk product previously defined may be used to solve the following system of n equations:

$$A_{11}\, x_1 + A_{12}\, x_2 + A_{13}\, x_3 + \ldots + A_{1n}\, x_n = 0 \qquad (29)$$
$$A_{21}\, x_1 + A_{22}\, x_2 + A_{23}\, x_3 + \ldots + A_{2n}\, x_n = 0 \qquad (30)$$
$$\vdots$$
$$A_{n-1,1} x_1 + A_{n-1,2} x_2 + A_{n-1,3}\, x_3 + \ldots + A_{n-1,n} x_n = 0 \qquad (31)$$
$$x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = 1 \qquad (32)$$

Coefficients of the i-th equation will be considered equal to the n components of vector $\mathbf{V}_1$, that is,

$$\text{Components of } \mathbf{V}_i = A_{i1},\ A_{i2},\ A_{i3},\ \ldots,\ A_{in} \qquad (33)$$

It is clear that if we perform the asterisk product of the (n - 1) vectors constructed with the first (n - 1) equations, the components of the resultant normalized vector will yield the solution of the system. As a numerical example let us solve the following problem:

$$x_1 - 2\, x_2 + 7x_3 - 4x_4 = 0 \qquad (34)$$
$$3x_1 + 6x_2 - 5\, x_3 + 8x_4 = 0 \qquad (35)$$
$$9x_1 - 10x_2 + 12x_3 - 11x_4 = 0 \qquad (36)$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \qquad (37)$$

The three vectors with 4 components each are:

$$\mathbf{V}_1 = [\ 1\ \ -2\ \ 7\ \ -4\ ] \qquad (38)$$
$$\mathbf{V}_2 = [\ 3\ \ 6\ \ -5\ \ 8\ ] \qquad (39)$$
$$\mathbf{V}_3 = [\ 9\ \ -10\ \ 12\ \ -11] \qquad (40)$$

Matrix [M], formed with vector {1} and V_i, i = 1,2,3 is:

$$[M] = [\{1\}\ V_1\ V_2\ V_3] = \begin{bmatrix} 1 & 1 & 3 & 9 \\ 1 & -2 & 6 & -10 \\ 1 & 7 & -5 & 12 \\ 1 & -4 & 8 & -11 \end{bmatrix} \qquad (41)$$

Matrix [S], obtained with the matrix product $[M]^T [M]$ is:

$$[S] = \begin{bmatrix} 4 & 2 & 12 & 0 \\ 2 & 70 & -76 & 157 \\ 12 & -76 & 134 & -181 \\ 0 & 157 & -181 & 446 \end{bmatrix} \qquad (42)$$

Vector C listing the first row cofactors of determinant $|S|$ is:

$$C \text{ transpose} = [\ 330532\ \ -119754\ \ -89810\ \ \ 5708\ ] \qquad (43)$$

The numerical value of determinant S is 4900 and the asterisk product involving vectors V_1, V_2 and V_3 is:

$$V_4 = V_1 * V_2 * V_3 = [M] \frac{C}{\sqrt{\det S}} = \begin{bmatrix} -104 \\ -370 \\ 140 \\ 404 \end{bmatrix} \qquad (44)$$

Finally, normalizing V_4, that is, dividing its components by $\sqrt{330532}$, the vector of solutions is found as follows:

$$\mathbf{X} = \begin{bmatrix} -0.180895 \\ -0.0643569 \\ 0.243513 \\ 0.702708 \end{bmatrix} \tag{45}$$

Another application of the numerical procedures described above is the generation of an orthogonal basis related to a pair of vectors in n-space. For instance, consider vectors **U** and **V** with components (4, -5, 7, 3) and (-11, 9, 3, 2), respectively. In the first example of this work we detailed the calculation of vector **W** as the result of the cross-product **U** x **V**. Performance of the asterisk product **U** * **V** * **W** leads to a fourth vector **P** and the normalized components of **U**, **V**, **W**, and **P** may be used to identify an orthonormal matrix. The reader may check that, to a few digit accuracy, normalized **P** has the following components: (-0.07606, -0.11197, -0.20943, 0.39297)

Conclusions

A simple matrix expression has been developed to generalize the definition of cross-product involving vectors with n components. When n>3 and two linearly independent vectors **U** and **V** are given, the new formulation yields a third vector W that satisfies equations 46 and 47.

$$\mathbf{U}.\mathbf{W} = 0, \ \mathbf{V}.\mathbf{W} = 0 \tag{46}$$

$$|\mathbf{W}|^2 = |\mathbf{U}|^2 |\mathbf{V}|^2 - (\mathbf{U}.\mathbf{V})^2 \tag{47}$$

These conditions coincide with those related to the conventional cross-product definition when n = 3. However, when n > 3, more than two linearly independent vectors, $\mathbf{V}_1$ thru $\mathbf{V}_m$ (m < n), can be given and a new vector **W** orthogonal to each of them might be of interest. The generalization established in this work leads to such a vector **W** which identifies the normal to the hyper-plane spanned by the given vectors. To avoid confusion with the multiple cross-product involving more than two vectors in a 3-space, another word should perhaps be used to describe the type of operations related to the generalized definition of the cross-product involving more than two vectors with n components each of them. The appellation asterisk product is suggested in this work.

It is clear that with the described numerical procedures we can generate a normal basis in n-space from two linearly independent vectors by means of a formulation different from the one used in the Gram-Schmidt orthogonalization method.

Finally, given (n - 1) vectors in an n-space, the asterisk product defined in this work yields a non-zero vector that is normal to each of them. Accordingly, such type of product may be considered the proper generalization of the cross-product.

The scalar product of two vectors with arbitrary number of components is used to test the orthogonality condition. The equation U.V=0 indicates orthogonal relationship. Similarly, after the generalized formulation of the cross-product herein presented, we may test the parallelism between vectors. A zero result for the cofactor C_1, as shown in equation (48), expresses the parallel relationship.

$$|\mathbf{U}|^2 \; |\mathbf{V}|^2 - (\mathbf{U} \cdot \mathbf{V})^2 = 0 \tag{48}$$