

Implementation of the stiffness method within MathCad numeric computation environment

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Abstract

This article presents a case study of a simple planar framed structure to show an original implementation of the stiffness method of structural analysis within MathCad numeric computation environment. This approach was used to introduce undergraduate students to the finite element method basis through the study of bar elements. As MathCad is a graphic environment, the student is able to 'see' the elements of the stiffness matrix, the effect of rotating from local to global coordinates, the assembling process of the structure stiffness matrix, and the decomposition used to introduce the boundary conditions. This approach has proven to be useful and didactic.

Sinopsis

Implantación del método de rigidez en el ambiente de computación numérico de MathCad

Este artículo presenta el estudio de un pórtico plano simple para mostrar una implantación original del método de rigidez para análisis estructural en el entorno de computación numérica provisto por MathCad. Este procedimiento se utilizó para familiarizar un grupo de alumnos subgraduados con las bases del método de elementos finitos a través del estudio de elementos de barra. Como MathCad es un ambiente gráfico, los estudiantes pueden 'ver' la matriz de rigidez de los elementos, el efecto de rotar de coordenadas locales a coordenadas globales, el proceso de ensamblado de la matriz de rigidez global de la estructura y la descomposición que se realiza para presentar las condiciones de contorno. Este enfoque ha demostrado ser muy útil y didáctico.

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Introduction

The stiffness method for the analysis of framed structures could be seen as a special case of the more general finite element method (FEM). The procedures used to derive the element stiffness matrix for any other finite element can be followed to obtain stiffness matrix or bar elements, which result in the same matrix that can be obtained from the direct stiffness method analysis. The peculiarity is that the shape functions used to interpolate the displacements within the element are the exact solution for the bar element (considering small displacements, and that the material is in the linear elastic range). One consequence of this situation is that in structures composed by bar elements the accuracy of the solution is not improved if the bars are subdivided in several finite elements.

Figure 1 shows a bar element, the local system of coordinates x_1, y_1 , and the degrees of freedom (DOF) related to each node. Each node can experiment three movements: two displacements and one rotation; thus a node has three DOF, and the element has six DOF. They are referred as generalized displacements, and denoted by a vector of six components $\{q_l\}$.

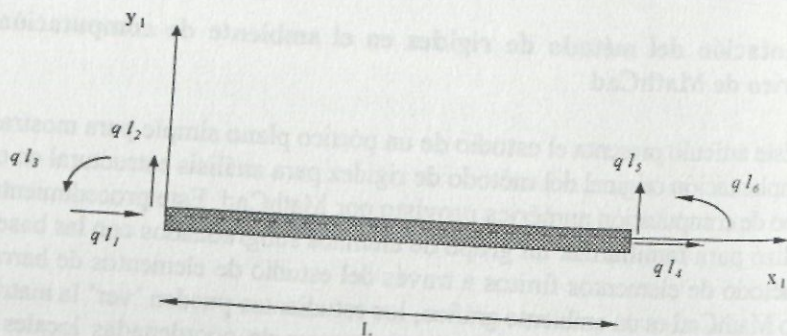


Figure 1. Bar element and local coordinate system

Applying any of the variational principles, i.e. the principle of virtual work, the equation relating the generalized member nodal forces $\{f\}$ and the generalized member nodal displacements $\{q\}$ may be obtained as follows:

$$[k] \{q\} = \{f\} \quad (1)$$

where $[k]$ is the member stiffness matrix in the local coordinate system, and is denoted by equation (2)

$$[k] = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{6EI}{L^2} & \frac{2EI}{L} \\ \frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \quad (2)$$

and

- L = member length
- A = cross sectional area
- I = moment of inertia of the cross section about the z axis
- E = Young's modulus of the bar material

In order to assemble the contribution of a member to the global response of the structure, the member's behavior has to be expressed in the global

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coordinate system: the generalized member nodal forces $\{fg\}$, the generalized member nodal displacements $\{qg\}$, and the member stiffness matrix $[kg]$ have to be rotated from local to global coordinates. Denoting by x_s and y_s the coordinates of the start node of the element, and x_e and y_e the coordinates of the end node of the member (fig. 2), the rotation is performed as follows:

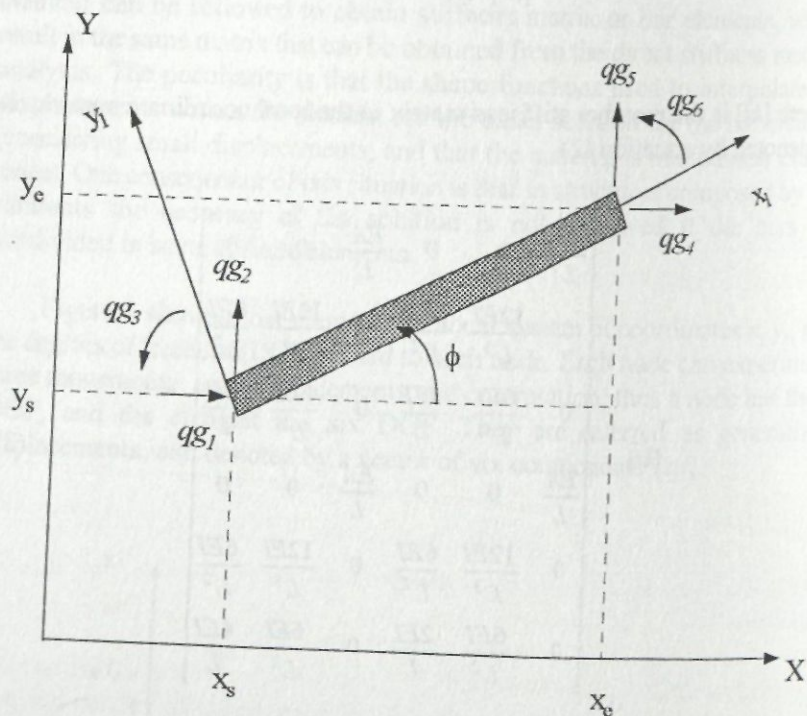


Figure 2: Bar element and global coordinate system

$$[k/l] \{q/l\} = \{f/l\} \quad (3)$$

$$[kg] = [T]^T [k/l] [T] \quad (4)$$

$$\{q_l\} = [T] \{q_g\} \quad (5)$$

$$\{f_l\} = [T] \{f_g\} \quad (6)$$

where:

$[kg]$ = member stiffness matrix in the global coordinate system

$[T]$ = transformation matrix, to rotate from global coordinate system to local coordinate system (eq. 7)

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 & 0 & 0 \\ -\sin\phi & \cos\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\phi & \sin\phi & 0 \\ 0 & 0 & 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (7)$$

and ϕ is the angle of inclination of the local coordinate system in respect to the global coordinate system. The expression of matrix $[T]$ as a function of the member nodes coordinates can be derived with equations (8) to (14).

Once the element stiffness matrix is rotated, it can be assembled to the structure stiffness matrix. A general form of the system equilibrium equation is obtained by assembling the contribution of each element to the global structure response.

$$\Delta x = x_e - x_s \quad (8)$$

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$$\Delta y = y_e - y_s \quad (9)$$

where

$$[KS]\{Q\} = \{F\} \quad (14)$$

$$L = \sqrt{\Delta x^2 + \Delta y^2} \quad (10)$$

$$\cos \phi = \lambda = \frac{\Delta x}{L} \quad (11)$$

$$\sin \phi = \mu = \frac{\Delta y}{L} \quad (12)$$

$$[T] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (13)$$

[KS] = structure stiffness matrix
 {Q} = vector of generalized nodal displacements

(displacements and rotations)
 $\{F\}$ = vector of generalized nodal loads (forces and moments)

The possible nodal movements are known as the degrees of freedom of the structure (DOF). In order to introduce the boundary conditions efficiently, it is recommended to enumerate the DOF according to the following procedure:

- a. Enumerate the nodes of the structure
- b. Enumerate (in a correlative way) only the DOF that are free to move, shifting from the first node to the last node in an ascending order, considering first the movement in the x direction, then the movement in the y direction, and last the rotation
- c. Repeat step b) enumerating the DOF that are restrained by supports (boundary conditions).

In this way, the following partition of the stiffness matrix may be performed, as equation (15) shows.

$$\begin{bmatrix} [K_{ff}] & [K_{fs}] \\ [K_{sf}] & [K_{ss}] \end{bmatrix} \begin{Bmatrix} Q_f \\ Q_s \end{Bmatrix} = \begin{Bmatrix} F_f \\ F_s \end{Bmatrix} \quad (15)$$

where:

- $\{Q_f\}$ = generalized displacements that are 'free' to move (unknown)
- $\{Q_s\}$ = generalized displacements that are 'supported' or restrained (known). In general they are zero; but may be different from zero to consider support settlement
- $\{F_f\}$ = generalized applied nodal loads, or external actions (known)
- $\{F_s\}$ = generalized nodal loads related to the supports, or reactions (unknown)

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According to this partition, the solution of the problem is performed in two stages:

- a. Applying equation (16) to obtain the unknown displacements $\{Q_f\}$
- b. Applying equation (17) to obtain the support reactions $\{F_s\}$

$$[K_{ff}] \{Q_f\} = \{F_f\} - [K_{fs}] \{Q_s\} \quad (16)$$

$$\{F_s\} = [K_{sf}] \{Q_s\} + [K_{ss}] \{Q_s\} \quad (17)$$

Once the nodal displacements are known, they can be used to obtain the member end forces by multiplying the member stiffness matrix by the corresponding member end displacements. Further comments on the stiffness method approach are presented within the implementation of the MathCad problem solution

Case study

The objective of the following problem is to analyze the framed structure shown in figure 3 by means of the stiffness method. All the members have the same material and section properties. The structure is subjected to punctual loads, and to a support settlement (support B settles 1 in).

The problem is solved in a step by step approach in order to show how the member stiffness matrix is computed and how it is assembled in the global stiffness matrix. The first step in the solution process is to enumerate the nodes and the members, as shown in figure 4.

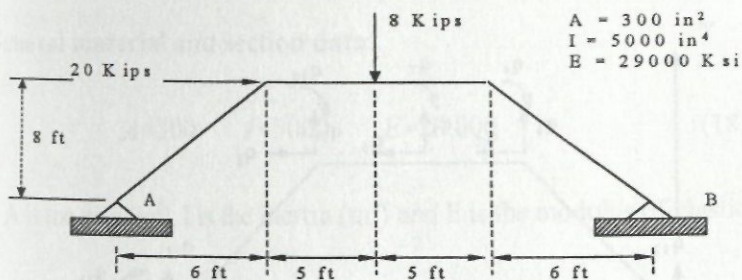


Figure 3: Frame geometry, properties, and applied load

- ⓪ Node Number
- ▭ Member Number

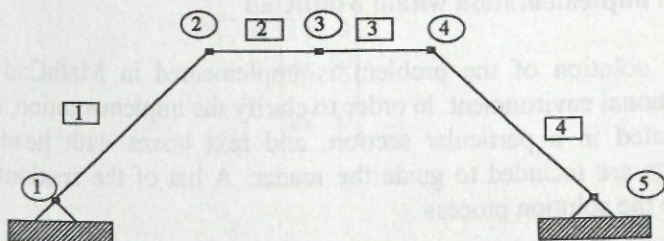


Figure 4. Node and member identification

Once the nodes are identified, the following step is to enumerate the nodal degrees of freedom (DOF). The frame has five nodes, totaling 15 DOF. Eleven DOF are free to move and four DOF are restrained by support conditions, as figure 5 shows.

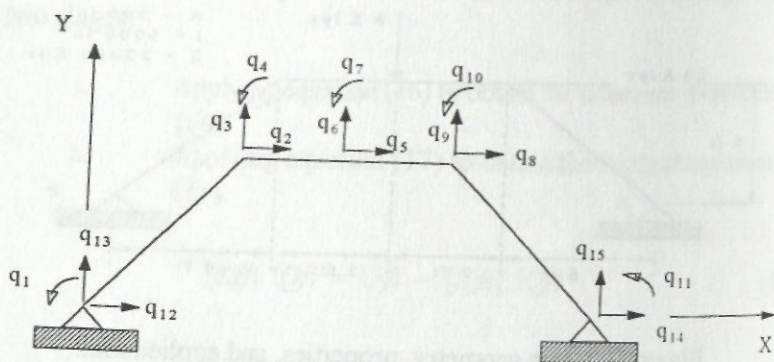


Figure 5: DOF identification

Solution implementation within MathCad

The solution of the problem is implemented in MathCad numeric computational environment. In order to clarify the implementation, each step is separated in a particular section, and text boxes with headings and comments are included to guide the reader. A list of the sections used to structure the solution process.

- I. Model description
- II. Matrices initialization and definition
- III. Member stiffness matrices and assembling process
- IV. Stiffness matrix partition
- V. Vector of generalized loads
- VI. Vector of nodal displacements
- VII. Vector of support reactions
- VIII. Vector of member end forces

1) Model description

1.1. General material and section data

$$A=300 \quad I=5000 \quad E=29,000 \quad (18)$$

where A is the area (in²), I is the inertia (in⁴) and E is the modulus of elasticity (Ksi)

1.2. Matrix of nodal coordinates

$$NC = \begin{bmatrix} 0 & 0 \\ 6 & 8 \\ 11 & 8 \\ 16 & 8 \\ 22 & 0 \end{bmatrix} \quad (19)$$

where

NC(i,1) = x coordinate of node i (in)

NC(i,2) = y coordinate of node (in)

1.3. Matrix of member connectivities

Where

MC(i,1) = start node of element i

MC(i,2) = start node of element i

$$MC = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \quad (20)$$

1.4. Matrix of nodal degree of freedom

$$NDOF := \begin{bmatrix} 12 & 13 & 1 \\ 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \\ 14 & 15 & 11 \end{bmatrix} \quad (21)$$

Where

$NDOF(i,1)$ = translational DOF in the x direction of node i

$NDOF(i,2)$ = translational DOF in the y direction of node i

$NDOF(i,3)$ = translational DOF in the z direction of node i

$$FDOF := 11, TDOF := 15 \quad (22)$$

DOF free to move and total numbers of DOG (free + restrained)

II. Matrices initialization and definitions

II.1 Initialization of variables

$$K_{TDOF, TDOF} := 0 \quad (23)$$

$$K_{FDOF, FDOF} := 0 \quad (24)$$

$$K_{FDOF, TDOF - FDOF} := 0 \quad (25)$$

$$K_{TDOF - FDOF, FDOF} := 0 \quad (26)$$

$$K_{TDOF - FDOF, TDOF - FDOF} := 0 \quad (27)$$

$$F_{FDOF} := 0 \quad (28)$$

$$U_{TDOF - FDOF} := 0 \quad (29)$$

where

K_s = complete structure stiffness matrix

F = vector of generalized nodal loads

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U_s = vector of support displacement

II.2. Member stiffness matrix in local coordinates and rotation matrix

$$KL(E, A, I, L) := \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 0 & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 0 & 6\frac{EI}{L^2} & 4\frac{EI}{L} & 0 & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix} \quad (30)$$

$$T(\lambda, \mu) := \begin{bmatrix} \lambda & -\mu & 0 & 0 & 0 & 0 \\ \mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\mu & 0 \\ 0 & 0 & 0 & \mu & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (31)$$

III Member stiffness matrices and assembling process

III.1 Member 1

$$NM := 1 \quad (32)$$

$$Ni := MC_{NM,1} \quad (33)$$

$$Nj := MC_{NM,2} \quad (34)$$

$$\Delta x := NC_{Nj,1} - NC_{Ni,1} \quad (35)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (36)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (37)$$

$$\lambda := \frac{\Delta x}{L} \quad (38)$$

$$\mu := \frac{\Delta y}{L} \quad (39)$$

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where

N_i = start node

$$\Delta x = 72, \Delta y = 96, L = 120, \lambda = 0.6, \mu = 0.8 \quad (40)$$

N_j = end node

Δx = x distance from start node to end node

Δy = y distance from start node to end node

L = member length

λ, μ = member orientation

$$KL(E, A, I, L) := \begin{bmatrix} 7.25 \cdot 10^4 & 0 & 0 & -7.25 \cdot 10^4 & 0 & 0 \\ 0 & 1.00694 \cdot 10^3 & 6.04167 \cdot 10^4 & 0 & -1.00694 \cdot 10^3 & 6.04167 \cdot 10^4 \\ 0 & 6.041674 \cdot 10^4 & 4.83333 \cdot 10^6 & 0 & -6.04167 \cdot 10^3 & 2.41667 \cdot 10^6 \\ -7.25 \cdot 10^4 & 0 & 0 & 7.25 \cdot 10^4 & 0 & 0 \\ 0 & -1.00694 \cdot 10^3 & -6.04167 \cdot 10^4 & 0 & 1.00694 \cdot 10^3 & -6.04167 \cdot 10^4 \\ 0 & 6.041674 \cdot 10^4 & 2.41667 \cdot 10^6 & 0 & -6.04167 \cdot 10^4 & 4.83333 \cdot 10^6 \end{bmatrix} \quad (41)$$

$$KG := T((\lambda, \mu) \cdot KL(E, A, I, L) \cdot T(\lambda, \mu)^T \quad (42)$$

KG is the member stiffness matrix in global coordinates

$$\begin{matrix}
 \text{KG} & \begin{bmatrix}
 2.67444 \cdot 10^4 & 3.43167 \cdot 10^4 & -4.83333 \cdot 10^4 & -2.67444 \cdot 10^4 & -3.43167 \cdot 10^4 & -4.83333 \cdot 10^4 \\
 3.43167 \cdot 10^4 & 4.67625 \cdot 10^4 & 3.625 \cdot 10^4 & -3.43167 \cdot 10^4 & -4.67625 \cdot 10^4 & 3.625 \cdot 10^4 \\
 -4.83333 \cdot 10^4 & 3.625 \cdot 10^4 & 4.83333 \cdot 10^4 & 4.83333 \cdot 10^4 & -3.625 \cdot 10^4 & 2.31667 \cdot 10^6 \\
 -2.67444 \cdot 10^4 & -3.43167 \cdot 10^4 & 4.83333 \cdot 10^4 & 2.67444 \cdot 10^4 & 3.43167 \cdot 10^4 & 4.83333 \cdot 10^4 \\
 -3.43167 \cdot 10^4 & -4.67625 \cdot 10^4 & -3.625 \cdot 10^4 & 3.43167 \cdot 10^4 & 4.67625 \cdot 10^4 & -3.625 \cdot 10^4 \\
 -4.83333 \cdot 10^4 & 3.625 \cdot 10^4 & 2.41667 \cdot 10^4 & 4.83333 \cdot 10^4 & -3.625 \cdot 10^4 & 4.83333 \cdot 10^4
 \end{bmatrix}
 \end{matrix} \quad (43)$$

Identification vector, showing the correspondence between the six-member DOF and the structure DOF

$$i := 1..3 \quad (44)$$

$$IDV_i := NDOF_{N_i, i} \quad (45)$$

$$IDV_{i+3} := NDOF_{N_j, i} \quad (46)$$

$$IDV := \begin{bmatrix}
 12 \\
 13 \\
 1 \\
 2 \\
 3 \\
 4
 \end{bmatrix} \quad (47)$$

Assembling process

$$i := 1..6 \quad (48)$$

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$$j := 1..6 \quad (49)$$

$$KS_{IDV, IDV_j} := KS_{IDV, IDV_j} + KG_{i,j} \quad (50)$$

Up to this moment, the structure stiffness matrix (KS) has the following elements:

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	$4.833 (10)^6$	$4.833 (10)^4$	$-3.625 (10)^4$	$2.417 (10)^6$	0	0	0	0	0	0	0	$-4.8333 (10)$	$3.625 (10)$
2	$4.833 (10)^4$	$2.674 (10)^4$	$-3.432 (10)^4$	$4.833 (10)^4$	0	0	0	0	0	0	0	$-2.674 (10)^4$	$-3.432 (10)$
3	$-3.625 (10)^4$	$3.432 (10)^4$	$4.676 (10)^4$	$-3.625 (10)^4$	0	0	0	0	0	0	0	$-3.432 (10)^4$	$-4.676 (10)$
4	$2.417 (10)^6$	$4.833 (10)^4$	$-3.625 (10)^4$	$4.833 (10)^6$	0	0	0	0	0	0	0	$-4.833 (10)^4$	$3.625 (10)$
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0	0
12	$-4.833 (10)^4$	$-2.674 (10)^4$	$-3.432 (10)^4$	$-4.833 (10)^4$	0	0	0	0	0	0	0	$2.674 (10)^4$	$3.432 (10)$
13	$3.625 (10)^4$	$3.432 (10)^4$	$-4.676 (10)^4$	$3.625 (10)^4$	0	0	0	0	0	0	0	$3.432 (10)^4$	$4.676 (10)$
14	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0

III.2 Member 2

$$NM := 2 \quad (52)$$

$$Ni := MC_{NM,1} \quad (53)$$

$$Nj := MC_{NM,2} \quad (54)$$

$$\Delta x := NC_{Nj,1} - NC_{Ni,1} \quad (55)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (56)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (57)$$

$$\lambda := \frac{\Delta x}{L} \quad (58)$$

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$$\mu := \frac{\Delta y}{L} \quad (59)$$

$$KG := T((\lambda, \mu)) \cdot KL(E, A, I, L) \cdot T(\lambda, \mu)^T \quad (60)$$

$$i := 1..3 \quad (61)$$

$$IDV_i := NDOF_{N_i, i} \quad (62)$$

$$IDV_{i+3} := NDOF_{N_j, i} \quad (63)$$

$$i := 1..6 \quad (64)$$

$$j := 1..6 \quad (65)$$

$$KS_{IDV_i, IDV_j} := KS_{IDV_i, IDV_j} + KG_{i, j} \quad (66)$$

III.3 Member 3

$$NM := 3 \quad (67)$$

$$Ni := MC_{NM,1} \quad (68)$$

$$Nj := MC_{NM,2} \quad (69)$$

$$\Delta x := NC_{Ni,1} - NC_{Nj,1} \quad (70)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (71)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (72)$$

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$$\lambda := \frac{\Delta x}{L} \quad (73)$$

$$\mu := \frac{\Delta y}{L} \quad (74)$$

$$KG := T((\lambda, \mu)) \cdot KL(E, A, I, L) \cdot T(\lambda, \mu)^T \quad (75)$$

$$i := 1..3 \quad (76)$$

$$IDV_i := NDOF_{N_i, i} \quad (77)$$

$$IDV_{i+3} := NDOF_{N_j, i} \quad (78)$$

$$i := 1..6 \quad (79)$$

$$j := 1..6 \quad (80)$$

$$KS_{IDV_r, IDV_j} := KS_{IDV_r, IDV_j} + KG_{i,j} \quad (81)$$

III.4 Member 4

$$NM := 4 \quad (82)$$

$$Ni := MC_{NM,1} \quad (83)$$

$$Nj := MC_{NM,2} \quad (84)$$

$$\Delta x := NC_{Nj,1} - NC_{Ni,1} \quad (85)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (86)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (87)$$

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$$NM := 4 \quad (82)$$

$$Ni := MC_{NM,1} \quad (83)$$

$$Nj := MC_{NM,2} \quad (84)$$

$$\Delta x := NC_{Nj,1} - NC_{Ni,1} \quad (85)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (86)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (87)$$

$$\lambda := \frac{\Delta x}{L} \quad (88)$$

$$\mu := \frac{\Delta y}{L} \quad (89)$$

$$KG := T((\lambda, \mu) \cdot KL(E, A, I, L) \cdot T(\lambda, \mu)^T \quad (90)$$

$$i := 1..3 \quad (91)$$

$$IDV_i := NDOF_{N_i, i} \quad (92)$$

$$IDV_{i+3} := NDOF_{N_j, i} \quad (93)$$

$$i := 1..6 \quad (94)$$

$$j := 1..6 \quad (95)$$

$$KS_{IDV_i, IDV_j} := KS_{IDV_i, IDV_j} + KG_{ij} \quad (96)$$

BIBLIOTECA

IV) Stiffness matrix partition

$$i := 1..FDOF \quad (97)$$

$$j := 1..FDOF \quad (98)$$

$$K_{ff_{ij}} := KS_{ij} \quad (99)$$

$$j := FDOF + 1..TDOF \quad (100)$$

$$K_{fs_{ij - FDOF}} := KS_{ij} \quad (101)$$

$$i := FDOF + 1..TDOF \quad (102)$$

$$j := 1..FDOF \quad (103)$$

$$K_{sf_i - FDOF, j} := K_{S_{ij}} \quad (104)$$

$$j := FDOF + 1..TDOF \quad (105)$$

$$K_{SS_{i - FDOF, j - FDOF}} := K_{S_{ij}} \quad (106)$$

V) Vector of generalized loads

$$F_2 := 20 \quad (107)$$

$$F_6 := -8 \quad (108)$$

$$Us_4 := 0 \quad (109)$$

$$F := F - K_{fs} \cdot Us \quad (110)$$

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VI Vector of nodal displacements

$$U_f := K_{ff}^{-1} \cdot F \quad (111)$$

$$U_f := \begin{pmatrix} 0 & 1 \\ 1 & -1.35066(10)^{-4} \\ 2 & 0.00817 \\ 3 & -0.00601 \\ 4 & 1.65786(10)^{-5} \\ 5 & 0.00807 \\ 6 & -0.00128 \\ 7 & 1.27596(10)^{-4} \\ 8 & 0.00798 \\ 9 & 0.00569 \\ 10 & 5.80486(10)^{-5} \\ 11 & -1.51436(10)^{-4} \end{pmatrix} \quad (112)$$

VII Vector of support reactions

$$R := K_{sf} \cdot U_f + K_{ss} \cdot U_s \quad (113)$$

$$R := \begin{bmatrix} -6.27198 \\ -3.27273 \\ -13.72802 \\ 11.27273 \end{bmatrix} \quad (114)$$

VIII Vector of member end forces

$$i := 1.. FDOF \quad (115)$$

$$U_i := Uf_i \quad (116)$$

$$i := 1.. TDOF - FDOF \quad (117)$$

$$U_{i + FDOF} := Us_i \quad (118)$$

VIII.1 Member 1

$$NM := 1 \quad (119)$$

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$$N_i := MC_{NM,1} \quad (120)$$

$$N_j := MC_{NM,2} \quad (121)$$

$$\Delta x := NC_{Nj,1} - NC_{Ni,1} \quad (122)$$

$$\Delta y := NC_{Nj,2} - NC_{Ni,2} \quad (123)$$

$$L := \sqrt{\Delta x^2 + \Delta y^2} \quad (124)$$

$$\lambda := \frac{\Delta x}{L} \quad (125)$$

$$\mu := \frac{\Delta y}{L} \quad (126)$$

$$KG := T(\lambda, \mu) \cdot KL(E, A, I, L) \cdot T(\lambda, \mu)^T \quad (127)$$

$$i := 1..3 \quad (128)$$

$$IDV_i := NDOF_{N_i, i} \quad (129)$$

$$IDV_{i+3} := NDOF_{N_j, i} \quad (130)$$

$$i := 1..6 \quad (131)$$

$$UM_i := U_{IDV_i} \quad (132)$$

$$FMG := KG \cdot UM \quad (133)$$

Then, FMG is the vector of nodal end forces in global coordinates

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$$FMG := \begin{bmatrix} -6.27198 \\ -3.27273 \\ 1.27898(10)^{-13} \\ 6.27198 \\ 3.27273 \\ 366.47324 \end{bmatrix} \quad (134)$$

$$FMLG := T(\lambda, \mu)^T \cdot FMG \quad (135)$$

and FML is the vector of nodal end forces, in local coordinates.

$$FML := \begin{bmatrix} -6.38137 \\ 3.05394 \\ 1.27898(10)^{-13} \\ 6.38137 \\ -3.05394 \\ 366.47324 \end{bmatrix} \quad (136)$$

Conclusions

The stiffness method has been successfully implemented within MathCad numeric computational environment. This implementation allows the analysis of simple framed structures. For a new structure, the model description has to be actualized (nodal coordinates, member connectivities, among others). Then, the contribution of each member may be added through a 'copy-paste'

procedure of the member stiffness matrix and assembling process, only actualizing the member identification number after each 'paste' editing operation within the computation sheet.

A major advantage of this implementation is that the student can actually 'see' the whole process: the member stiffness matrix in local and global coordinates; how each element contributes to the global structure stiffness matrix; how the partition of the matrix is performed to incorporate the boundary condition; how the member end forces are obtained once the nodal displacements are known.

This approach has been found to be useful to introduce the mechanical aspects of the stiffness method, as a special case of the more general finite element analysis method. Special conditions such as support settlement or elastic foundations can be easily incorporated in the analysis process, and constitute a good analytical case of study to be implemented by the students.

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