Inviscid, Incompressible CFD Solver with Coordinate Transformation for Aerodynamic Applications

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Abstract **—** *Fluid mechanics is a complex field of study with many modern design applications. Mechanical and aerospace engineers frequently have the need to analyze fluid flow patterns for practical design purposes ranging from a complete design to a validation of results. Physical concepts, such as conservation of mass, conservation of momentum, and conservation of energy are required to fully describe any arbitrary flow pattern's characteristics. These concepts are required to compute pressure distribution, typically used in the design of airfoils and other arbitrary shapes, which are of main interest in the aerospace and naval industries. A tool was created that employs computational fluid dynamics techniques to provide solution for flow patterns over NACA four digit airfoils. The tool uses a coordinate transformation method to analyze flow properties at desired points without the need of interpolation, which can affect accuracy of results. Additionally, the tool considers flows whose velocity potential conforms to Laplace's linear partial differential equation on a plane.*

Key Terms — *Computational Fluid Dynamics, Coordinate Transformation, Inviscid-Incompressible Flow, Laplace PDE.*

INTRODUCTION

The aerospace and naval industries require professionals who master fluid mechanics so that these may find solutions to their design problems in fluid dynamics. These professionals employ tools and solution methods that aid in the analysis and design process. Designing for aerospace and naval applications constantly call for the need to analyze aerodynamic properties of airfoils. This type of analysis is of a complex nature and many times provides for unreliable results. To serve this purpose, a software tool was created using Matlab that analyzes flow patterns over basic airfoils.

The main idea behind this program is that if the flow speed distribution is obtained some distance from the airfoil and afterwards, the pressure distribution around the airfoil's surface may also be obtained. An incompressible, inviscid flow is used to analyze the fluid flow patterns. A coordinate transformation technique is also employed to ensure that properties obtained at the airfoil's body is obtained, given that this is the main area of interest for aerodynamic applications.

GOVERNING THEORY

The model used for the flow pattern analysis in question is that of an incompressible, inviscid flow. This flow is unaffected by viscous effects and its density remains constant throughout the field.

To be more precise, an incompressible flow is any flow in which the particles' density remains constant. The density is the mass of the particle per unit volume, where ρ (rho) is the density, m is the mass, and V is the volume. If the mass of a particle was fixed and it would be moved through an incompressible flow, then this particle must also have a constant volume (that is $\int \delta V$ is equal to V, where V is constant) is to comply with the constant density condition. Mathematically, this is,

$$m = \int \rho dV \tag{1}$$

Now, as defined by Anderson Jr. [1], if the volume of the flow is to remain constant and the volume under question is sufficiently small so that it only contains one particle (that is, $V_{particle} = \delta V$), then it follows that the time rate of change of volume of a fluid element per unit volume should be as follows,

$$\frac{1}{\delta V}\frac{D(\delta V)}{Dt} = 0 \quad (2)$$

It then follows that the divergence of velocity must be equal to the time rate of change of volume of a fluid element per unit volume. We define u and v as the velocity vector components in the x axis and y axis, respectively. Now remember that the divergence of velocity is the derivative of all the velocity components with respect to their own direction. In an x-y plane this is,

$$\nabla \bullet V = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} = \frac{1}{\delta V}\frac{D(\delta V)}{Dt} = 0 \quad (3)$$

The second condition requires that the flow should be irrotational. This condition is given by computing the curl of the velocity vector field. For this case, the flow's curl should be equal to zero. The curl for the flow's velocity vector field is given by the alternate form of its derivative. This is,

$$\nabla \times V = \frac{\delta u}{\delta y} - \frac{\delta v}{\delta x} = 0 \quad (4)$$

If we were to define a function in so that its gradient defines the velocity of a flow field, we may expand our definition for any particle moving through an incompressible and irrotational flow. As Anderson1 named it, the velocity potential is denoted with the Greek letter ϕ (phi) and shall serve this purpose. If we were to combine this with the curl and the divergence of a vector field, we obtain the following,

$$\nabla \times (\nabla \varphi) = \frac{\delta}{\delta y}\left(\frac{\delta \varphi}{\delta x}\right) - \frac{\delta}{\delta x}\left(\frac{\delta \varphi}{\delta y}\right) = 0 \quad (5)$$

and,

$$\nabla \bullet (\nabla \varphi) = \nabla^2 \varphi = \frac{\delta^2 \varphi}{\delta x^2} + \frac{\delta^2 \varphi}{\delta y^2} = 0\,, \quad (6)$$

in cartesian coordinates

$$\nabla^2 \varphi = \frac{\delta}{\delta r}\left(r\frac{\delta \varphi}{\delta r}\right) - \frac{1}{r}\frac{\delta}{\delta \theta}\left(\frac{\delta \varphi}{\delta \theta}\right) = 0\,, \quad (7)$$

in cylindrical coordinates.

Note that the first equation only shows compliance with the irrotational condition and the second equation, which shows the incompressibility condition, is also Laplace's differential equation [2]. Anderson also defined that the streamlines (denoted as ψ) for any flow pattern under these conditions must also satisfy Laplace's equation. That is,

$$\nabla^2 \psi = 0 \quad (8)$$

Boundary Conditions in the Physical Plane

After solving Laplace's equation, the required boundary conditions are the following. The first condition is applied on the surface of the body being analyzed. This requires that the Kutta condition must be applied at the trailing edge [3]. This requires that the stream function values on the trailing edge of a finite airfoil are such that the trailing edge yields a stagnation point, or a point of zero velocity. The Kutta condition also requires that, at a given speed and angle of attack, the value of the circulation around the airfoil is such that the streamline leaves the airfoil smoothly, or ensure that the airfoil is a streamline of the flow. Given that the airfoil is a streamline of the flow, it also follows that the streamline value on the surface of the airfoil must be a constant value. Refer to Figure 1 for application of the Kutta condition on a finite airfoil. The second boundary condition applies to the values at the "borders" of the flow field. These conditions, known as the infinity boundary conditions, require that the stream values be such that the derivatives at a discrete point should yield the freestream speed at the prescribed angle of attack. These conditions will be reviewed later from the coordinate transformation point of view. The equations that represent these conditions in the physical coordinate system are the following,

$$u = \frac{dy}{dy} \quad (9)$$

$$u = -\frac{dy}{dx} \quad (10)$$

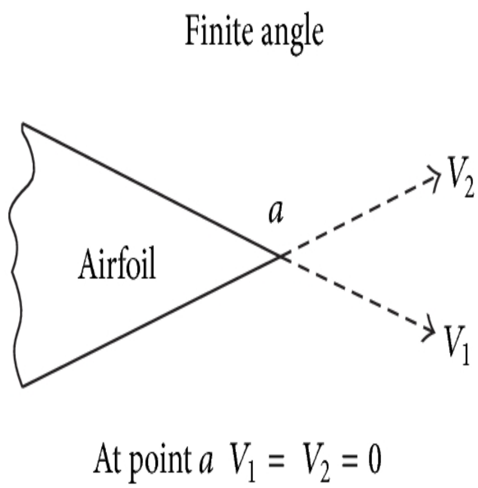

Figure 1
Application of the Kutta Condition

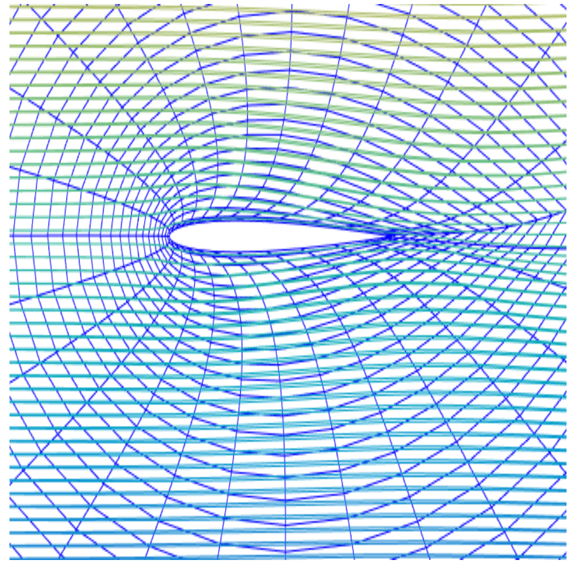

Figure 2
Contour Map of Coordinate Transformation Variables in Physical Plane

Coordinate Transformation into Computational Plane

To capture the physics when computing the fluid flow patterns, it is necessary to perform a coordinate transformation from the physical plane to a convenient computational plane. This requires that the Cartesian plane be transformed into a plane where the x axis and y axis are represented by two new variables.

Although the definition of the new variables is arbitrary by convenience, observation of the field calls for an imposed relation between the Cartesian coordinates and the computational coordinates. The imposed relationship arises from the desire to represent the airfoil "contour" with some constant value and the outer boundary "contour" with another representative value. The imposed relationship is called a boundary-fitted coordinate system, subjected to an elliptic relationship between the variables. Refer to Figure 2 for more details on coordinate transformation relationship.

Observe that the blue contours that surround the airfoil will represent the variable eta, η, and the blue contours that expand in a transverse direction to that of the airfoil will be represented by the variable zeta, ζ.

An arbitrary grid is created in such a way that it encloses the body being analyzed. Figure 3 shows an example on the physical plane with the intention of clarifying concept. The airfoil contour follows line *racegpr*, which represents the inner boundary for our grid generation method, represented by the constant value line, η_1. Line *sbdfhqs* represents the outer boundary, represented by the constant value line, η_2. Lines *pq* and *rs* are coincident for when applying the model, but separated in Figure 3 for illustrative purposes. Figure 4 demonstrates the computational plane and the respective locations of points *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *p*, *q*, *r*, and *s*. The lower horizontal line in the grid represents the body contour, whereas the upper horizontal line represents the outer boundary.

The simplest possible elliptic relationship between the physical coordinate variables and the computational variables is Laplace's linear partial differential equation [4]. Since we are interested in computing a grid with constant lower and upper boundaries, the appropriate applied conditions are Dirichlet boundary conditions. Notice that this only describes the upper and lower limits. Since this system envelops an entire physical grid, recall lines *pq* and *rs* in Figure 3, the values for Γ_3 and Γ_4 must be such that the physical coordinates along both contours be equal to each other, thus closing the physical plane and representing re-entrant boundary conditions [5]. The elliptic system relationship is analogous to ζ (x, y) and η (x, y) being harmonic in the physical plane. The following coupled system of equations is posed as the solution to the coordinate transformation,

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0 \quad (11)$$

$$\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} = 0 \quad (12)$$

subjected to the Dirichlet boundary conditions,

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi_1(x, y) \\ \eta_1 \end{bmatrix}, \; [x, y] \;\; \varepsilon \;\; \gamma_1 \quad (13)$$

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \xi_2(x, y) \\ \eta_2 \end{bmatrix}, \; [x, y] \;\; \varepsilon \;\; \gamma_2 \quad (14)$$

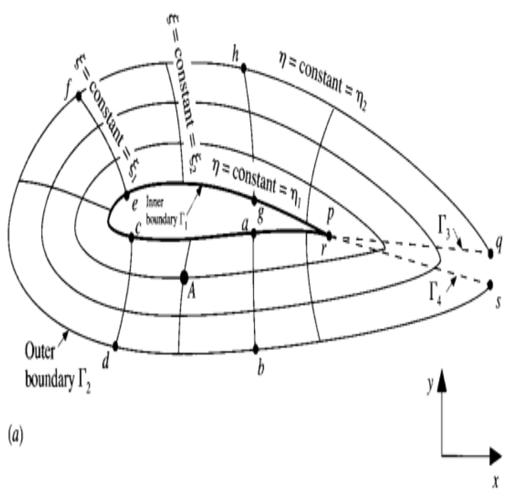

Figure 3
Physical Plane Concept for Arbitrary Body Shape Limits

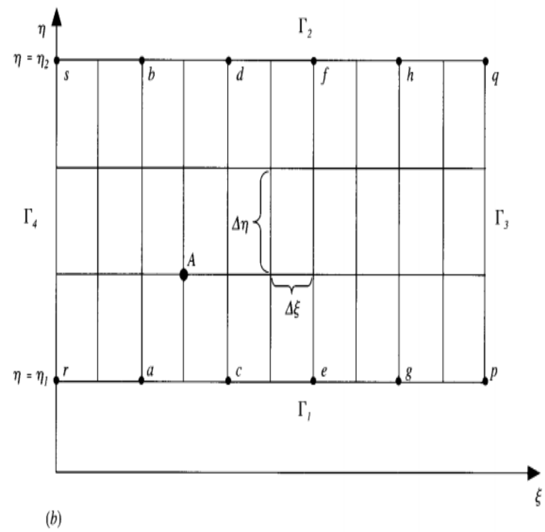

Figure 4
Physical Plane Concept for Arbitrary Body Shape Limits

Notice that the solution to this system yields the distribution of the computational variables in the physical plane. The dependent variables here are the physical coordinates and this will not yield the desired computational plane. Therefore, the desired computational plane must be imposed, such that the independent variables are the computational field variables. This condition then requires that the system of equations to be solved is not the Laplace partial differential equations, but their inverse, instead. The solution to this system will yield the distribution of physical coordinates required to hold the elliptic relationship. The development of this equation will not be presented, although the process may be found in many differential equations textbooks. The system of equations that must be solved are the following,

$$\frac{\alpha \partial^2 x}{\partial \xi^2} - \frac{2\beta \partial^2 x}{\partial \xi \partial \eta} + \frac{\gamma \partial^2 x}{\partial \eta^2} = 0 \quad (15)$$

$$\frac{\alpha \partial^2 y}{\partial \xi^2} - \frac{2\beta \partial^2 y}{\partial \xi \partial \eta} + \frac{\gamma \partial^2 y}{\partial \eta^2} = 0 \quad (16)$$

where,

$$\alpha \equiv \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \tag{17}$$

$$\beta \equiv \left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right) \tag{18}$$

$$\gamma \equiv \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 \tag{19}$$

subjected to the transformed boundary conditions,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f_1(\xi,\eta_1) \\ f_2(\xi,\eta_1) \end{bmatrix}, \ \lfloor \xi,\eta_1 \rfloor \ \varepsilon \ \ \gamma_1 * \tag{20}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} g_1(\xi,\eta_2) \\ g_2(\xi,\eta_2) \end{bmatrix}, \ \lfloor \xi,\eta_1 \rfloor \ \varepsilon \ \ \gamma_2 * \tag{21}$$

The functions on the boundary conditions correspond to the known shape of the contours in the physical coordinate plane. Since the horizontal limits represent re-entrant boundaries, boundary values are not necessary to carry out the solution.

The new system of equations is quasi-linear in nature [6] and is more complex to solve than Laplace's partial differential equation, but it complies with all the requirements for a proper coordinate transformation, along with fitting the physical contours required to capture the physical phenomena. Notice that the boundary conditions can easily incorporate multiple bodies, if desired.

Final Form of the Quasilinear Partial Differential Equation

No exact solution exists for the new system of equations, so approximation methods must be employed. To approximate our solution, second order finite difference techniques are employed, applied to all the corresponding derivatives. The f (x or y) coordinate derivative approximations are the following,

$$\left(f_\xi\right)_{i,j} = f_{i+1,j} - f_{i-1,j} \tag{22}$$

$$\left(f_\eta\right)_{i,j} = f_{i,j+1} - f_{i,j-1} \tag{23}$$

$$\left(f_{\xi\xi}\right)_{i,j} \approx f_{i+1,j} - 2f_{i,j} + f_{i-1,j} \equiv \left(f'_{\xi\xi}\right)_{i,j} \tag{24}$$

$$\left(f_{\eta\eta}\right)_{i,j} \approx f_{i,j+1} - 2f_{i,j} + f_{i,j-1} \equiv \left(f'_{\eta\eta}\right)_{i,j} \tag{25}$$

$$\left(f'_{\xi\eta}\right)_{i,j} \approx f_{i+1,j+1} - f_{i+1,j-1} + f_{i-1,j-1} - f_{i-1,j+1} \tag{26}$$

The approximations for f are analogous to x and y. Once these derivatives are developed, the difference approximations of the derivatives are constructed and yield the following $(I_{max}\text{-}1) \times (J_{max}\text{-}2)$ amount of equations,

$$x_{i,j} = \left[\alpha'_{i,j}\left(x_{i-1,j} + x_{i+1,j}\right) - \beta'_{i,j}\left(x'_{\xi\eta}\right)_{i,j}/2 + \gamma'_{i,j}\left(x_{i,j+1} + x_{i,j-1}\right)\right] / \left[2\left(\alpha'_{i,j} + \gamma'_{1,j}\right)\right] \tag{27}$$

$$y_{i,j} = \left[\alpha'_{i,j}\left(y_{i-1,j} + y_{i+1,j}\right) - \beta'_{i,j}\left(y'_{\xi\eta}\right)_{i,j}/2 + \gamma'_{i,j}\left(y_{i,j+1} + y_{i,j-1}\right)\right] / \left[2\left(\alpha'_{i,j} + \gamma'_{1,j}\right)\right] \tag{28}$$

where at $i=1$ the corresponding discrete points pertain to the re-entrant boundary conditions.

Solution Scheme of the System Quasilinear Partial Differential Equations

The solution to this system requires a technique that can solve non-linear partial differential equations. To serve this purpose, the Successive over Relaxation, Steffensen-Newton method is used.

This method is similar to the Gauss – Seidel iterative technique with an adapted version of the Newton – Raphson method. Research on this method states that this method offers faster convergence rates than other methods and will serve our computational purposes, given that both methods can be used for non-linear systems.

Since the equations are an approximation of a quasi-linear set of partial differential equations, initial estimate values are necessary to ensure proper solution values. The approximation offered for the initial conditions is a weighted average of boundary points and can be found in [5]. Observe Figure 5 for a plot of the initial guess pattern.

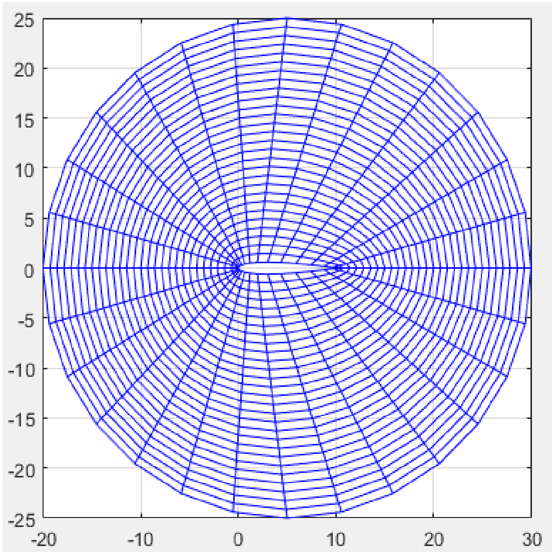

Figure 5
Initial Approximation Plot

The Successive Over Relaxation – Steffensen Newton method is applied as follows. The system of equations must be stated in the following format,

$$F_i(x_i) = 0 \tag{29}$$

The idea is that the entire system of equations be equal or approximate to zero until a certain convergence parameter is achieved. Convergence is reached when the maximum relative error between the variables is below the specified convergence factor. The updated variables are used immediately upon calculation, thus speeding the iterative process. Refer to [7] for instructions on how to apply the method.

Convergence of these equations yield the desired *x* and *y* coordinate distribution. Observe Figure 6 for the final contour plot for the same case as that in Figure 5.

CASE STUDY

An inviscid, incompressible flow's streamlines must conform to Laplace's partial differential equation. Coordinate transformation requires that any equation being solved also be transformed to the computational plane. The equation for the stream function on the transformed plane is as follows,

$$\frac{\alpha\partial^2\Psi}{\partial\xi^2} - \frac{2\beta\partial^2\Psi}{\partial\xi\partial\eta} + \frac{\gamma\partial^2\Psi}{\partial\eta^2} = 0 \tag{30}$$

subjected to the transformed version of Neumann boundary condition,

$$\Psi(\xi,\eta_1) = \Psi_0 \tag{31}$$

$$\Psi(\xi,\eta_2) = y(\xi,\eta_2)\cos(\theta) - x(\xi,\eta_2)\sin(\theta) \tag{32}$$

and imposition of the Kutta condition at the trailing edge,

$$V_{TE} = 0 \tag{33}$$

Contour plots of the solution of this system at a given velocity and angle of attack gives the streamline pattern for the flow over a selected airfoil.

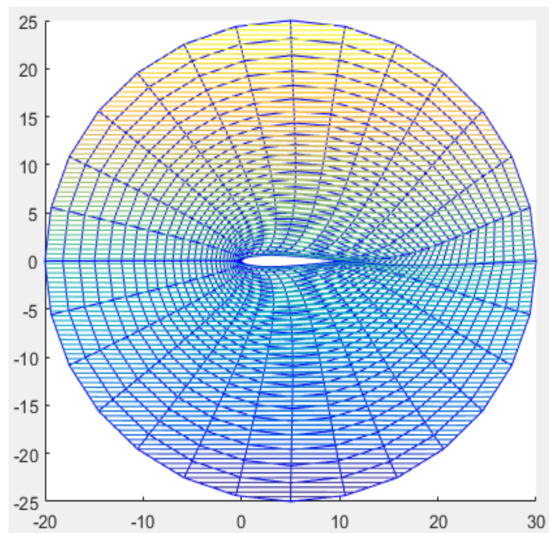

Figure 6
Final Contour Plot

Figures 7, 8, and 9 demonstrate the streamline patterns obtained for a NACA 0012 airfoil with a chord size of 10, freestream velocity of 100, convergence factor of 1E-12, and at angles of attack of 0, 15 degrees, and 45 degrees, respectively.

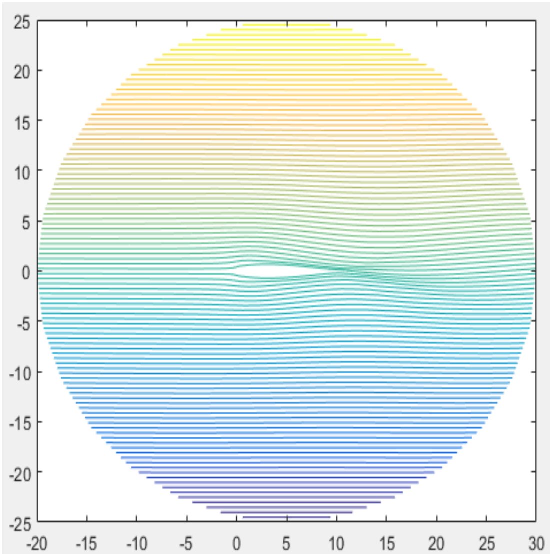

Figure 7
Angle of Attack of 0 Degrees

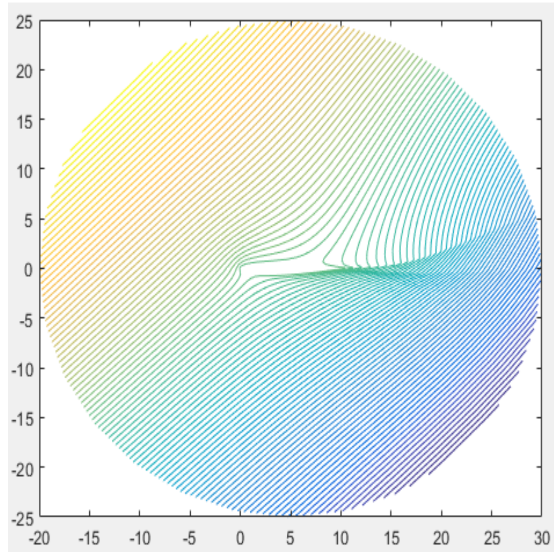

Figure 8
Angle of Attack of 45 Degrees

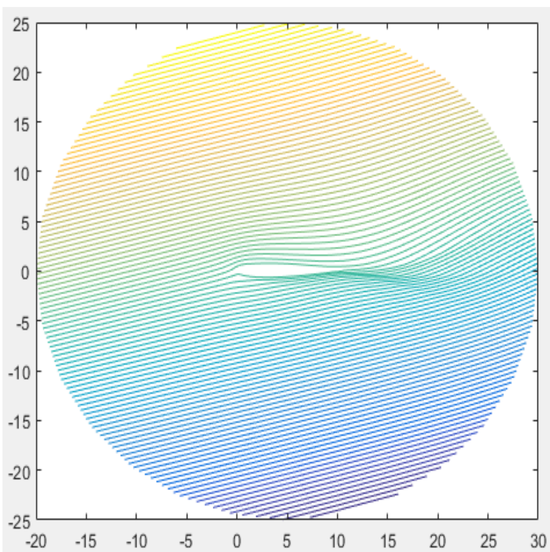

Figure 8
Angle of Attack of 15 Degrees

Notice that the stagnation points can be observed in all three plots. Close inspection of the contours shows that none of them cross each other, thus specifying that the flow patterns are correct. From these streamline patterns, other flow conditions may be obtained following the relationships of the transformed coordinate plane, thus ensuring that the desired physics are captured.

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