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Inscribed r-gons: A Combinatorial approach

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Abstract

The author uses the concept of circular combinations to find the number of *r-gons* that can be inscribed in a *n-gon*, being this number

$$\frac{n(n-r-1)!}{r!(n-2r)!}, \quad n \geq 2r, \quad n > 5, \quad r > 2.$$

When $r=2$, the above formula gives the number of diagonals of the *n-gon*.

R-ágonos inscritos: Acercamiento a base de combinaciones

Sinopsis

Se usa el concepto de combinaciones circulares para hallar el número de *r-ágonos* que se pueden inscribir en un *n-ágono*, siendo este número

$$\frac{n(n-r-1)!}{r!(n-2r)!}, \quad n \geq 2r, \quad n > 5, \quad r > 2.$$

Cuando $r=2$, la fórmula anterior da el número de diagonales del *n-ágono*.

1. Introduction

The central problem of combinatorial theory may be considered to be that of arranging objects according to specified rules, and finding out in how many ways the objects can be arranged. If the rules are simple, the emphasis is on enumerating of the ways in which the objects can be arranged. If, on the other hand, the rules are complicated, the chief problem is whether such arrangements do exist, and to find appropriate methods to construct them[1]

The basic, most powerful, and useful concepts of combinatorial theory, namely the permutations and combinations of n objects taken r at a time with or without repetitions, are based on the *Fundamental Principle of Enumeration*. Given n different objects, this principle allows us to find the total number of ways of filling r positions, $r \leq n$, with n given objects. The first position can be occupied by any of the n objects, so we have $(n-1)$ remaining ones for the second, $(n-2)$ for the third, and so on. Thus, the r^{th}

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[1] Mall, M., 1986, Combinatorial Theory, John Wiley & Sons, 2nd edition, NY

position can be occupied by any of the $(n-r+1)$ remaining objects. Since the choices for each position are arbitrary, in all we can have, $n(n-1)(n-2)...(n-r+1)$.

After a short discussion about linear permutations and linear combinations, we introduce the concept of circular combinations, which leads to the establishment of the formula to find the number of polygons that can be inscribed in another polygon. As a particular case we find the number of diagonals of a polygon of n sides.

2. Permutations and combinations

Definition 2.1.

A permutation is an ordered selection of objects from a set S.

Definition 2.2.

A combination is an unordered selection of objects from a set S.

In order to determine the number of permutations of n objects taken r at a time without repetitions, in symbols $P_{n,r}$, let the list $\{a_1, a_2, ..., a_r\}$ represent a generic permutation. In this way, we may choose a_1 to be any of the n given objects, a_2 any of the remaining $(n-1)$, and having chosen $\{a_1, a_2, ..., a_i\}$, $I < r$, we may take a_{i+1} as any of the $(n-I)$ remaining elements. Thus, by the Fundamental Principle of Enumeration

$$P_{n,r}=n(n-1)...(n-r+1)=\frac{n!}{(n-r)!} \quad (1)$$

Now, according to (1), a combination, $\{a_1, a_2, ..., a_r\}$, of r different objects will lead to $r!$ different permutations, namely all permutations of r elements taken r at a time without repetitions. In fact,

$$P_{r,r}=\frac{r!}{(r-r)!}=\frac{r!}{0!}=\frac{r!}{1}=r! \qquad (2)$$

Therefore, the number of combinations of a set of n objects taking r at a time without repetitions, in symbols $C_{n,r}$ or $\binom{n}{r}$, is given by

$$C_{n,r}=\binom{n}{r}=\frac{P_{n,r}}{r!}=\frac{n!}{r!(n-r)!} \qquad (3)$$

2.1 Exclusion of consecutive elements

To find the number of combinations of a set of n objects, $\{a_1, a_2, \ldots, a_n\}$, taken r at a time without repetitions, and excluding consecutive elements, we proceed as follows:

1st. Make a linear array with the given sequence of elements, $\{a_1, a_2, \ldots, a_n\}$.

2nd. Make a selection of r elements according to the specifications, and put a mark right after each selected element.

3rd. Count the number of elements before the first mark, between consecutive marks, and after the last mark.

4th. If there are x_1 elements before the first mark, x_j between the marks $(j-1)$ and j, $2 \le j \le r$, and x_{r+1} after the last mark, the set $\{x_1, x_2, \ldots, x_r\}$ determines the choice or *linear combination*, and

$$\Sigma_{i=1}^{r+1} x_i = n \qquad (4)$$

being $x_1 \geq 1, x_2 \geq 2, \ldots, x_r \geq 2, x_{r+1} \geq 0.$

Consequently, the number of solutions of (4) is equal to the number of linear combinations of the set of n objects $\{a_1, a_2, \ldots, a_n\}$ taken r at a time without repetitions, and excluding consecutive elements.

Equation (4) can now be conveniently modified, subtracting 1 from each x_i, $2 \leq I \leq r$, and adding 1 to x_{r+1}, to get a representation of $(n-r+2)$ as a sum of $(r+1)$ positive integers. Thus,

$$x_1 + (x_2 - 1) + \ldots + (x_r - 1) + (x_{r+1} + 1) = n - r + 2 \qquad (5)$$

or

$$\Sigma_{i=1}^{r+1} y_i = n - r + 2 \qquad (6)$$

being

$$y_i = \begin{cases} x_i & if \quad i=1 \\ x_i - 1 & if \quad 2 \leq i \leq r \\ x_i + 1 & if \quad i = r+1 \end{cases} \qquad (7)$$

and $y_i \geq 1$ for all I.

Hence, the number of *non-negative* integral solutions of (4) is equal to the number of *positive* integral solutions of (6), which can be seen as the number of ways of putting r marks in $(n-r+1)$ spaces, that is

$$\binom{n-r+1}{r} \tag{8}$$

Therefore, the number of linear combinations of a set $\{a_1, a_2, \ldots, a_n\}$, taken r elements at a time without repetitions, and excluding consecutive ones, is given by (8).

3. Circular combinations

Definition 3.1

If n elements are arbitrarily arranged on a circular, or simple closed path, any selection of r of them, $r \leq n$, is called a circular combination.

If we make a circular array with the elements of the set $\{a_1, a_2, \ldots, a_n\}$ in this order, the number of combinations (5) will be reduced by the number of combinations containing a_1 and a_n, because, in this array, these two elements are consecutive. Furthermore, since the combinations containing a_1 and a_n do not include a_2 and a_{n-1}, we have *(n-4)* remaining elements to take *(r-2)* at a time without repetitions, and excluding consecutive ones. Thus, proceeding as before, we can say that the number of combinations containing a_1 and a_n is

$$\binom{(n-4)-(r-2)+1}{r-2} \tag{9}$$

or

$$\binom{n-r-1}{r-2} \tag{10}$$

Therefore, the number of combinations of a set of n objects, $\{a_1, a_2, ... , a_n\}$, taken r at a time without repetitions, and excluding consecutive elements is

$$\binom{n-r+1}{r} - \binom{n-r-1}{r-2} = \tag{11}$$

$$\frac{(n-r+1)!}{r!(n-2r+1)!} - \frac{(n-r-1)!}{(r-2)!(n-2r-1)!} = \tag{12}$$

$$\frac{(n-r+1)! - r(r-1)(n-r-1)!}{r!(n-2r+1)!} = \tag{13}$$

$$\frac{(n-r-1)![n(n-r+1) - nr + r^2 - r - r^2 + r]}{r!(n-2r+1)(n-2r)!} = \tag{14}$$

$$\frac{n(n-r-1)!(n-2r+1)}{r!(n-2r+1)(n-2r)!} = \tag{15}$$

$$\frac{n(n-r-1)!}{r!(n-2r)!} \tag{16}$$

This result can be restated and used to find the number of polygons inscribed in another polygon.

Definition 3.2.

A polygon of r sides, or r-gon, A, is said to be inscribed in a polygon of n sides, or n-gon, B, if the vertices of A are non-consecutive vertices of B.

Theorem 3.1.

For $n \geq 2r$, $n > 5$, and $r > 2$, the number of inscribed r-gons in a n-gon is given by the formula

$$\frac{n(n-r-1)!}{r!(n-2r)!} \tag{17}$$

Proof: From *Definition 3.2* it follows that *r-gons* inscribed in *n-gons* exist if and only if:

- $r > 2$ (the simplest polygon is by definition a triangle)
- $n > 5$ (to inscribe a triangle we need at least a hexagon)
- $n \geq 2r$ (if $n < 2r$, we cannot choose r non-consecutive vertices)

For $n > 5$, make a circular array with the sequence $\{1, 2, \ldots, n\}$, and let the corresponding points be the vertices of a *n-gon*.

For $r > 2$, any combination of r non-consecutive vertices defines an inscribed *r-gon*.

Apply equations (11-16)

Corollary 3.1

The number of diagonals of a n-gon is equal to

$$\frac{n(n-3)}{2} \tag{18}$$

Proof

Use equation (16) with $r=2$.