3D Vibration Isolation Systems Modeling for Air Handling Units

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Abstract — *Spring isolators systems are commonly used in a variety of ways in today's construction industry. They are an essential part of the design of many mechanical systems that are in need of absorbing and dissipating unwanted vibrations. In the construction industry, vibration isolation systems have been installed without completely understanding how these systems are selected. The goal of this paper is to create a tool that will make the selection process easier for designers and installers. A Matlab© [1, 4] computer program will be created to give graphical models of the steady state vibrating motion of an Air Handling Unit (AHU) with a selected vibration isolation system. The program will be able to simulate any model with the phenomena of a vibrating rigid body on a resilient supported element which will be caused by a vibratory force, or moment, that is generated within the body, creating unbalance. This program will also be able to calculate the natural frequencies of the mode in order to make sure that it will not cause the model to fall under the unwanted effect of resonance that will eventually cause our model and its supports to fail.*

Key Terms — *Isolators, Natural Frequency, Resonance, Vibration Isolation.*

VIBRATION AND FREE RESPONSE

The physical explanation of the phenomena of vibration concerns with the interplay between potential energy and kinetic energy [2]. The degree of freedom of a system is the minimum number of displacement coordinates needed to represent the position of the system's mass at any instant of time [2]. Since we are dealing with a 3D system, our model contains 6 degrees of freedom. Free response refers to analyzing the vibration of a system resulting from a nonzero initial displacement and/or velocity of the system with no external force or moment applied [2].

For a mass and spring combination the equation of the force that is applied to the spring (f_k) in order to move the mass is lineally related to the compressed distance. For displacement in the x direction,

$$f_k = kx \tag{1}$$

where k is the spring stiffness and x is the displacement of the mass in the x direction. A spring of stiffness k will store the amount of $\frac{1}{2}ky^2$ as potential energy.

In the case of a mass and spring combination moving on a frictionless surface in a horizontal orientation and considering all the forces acting along the x-direction yields to,

$$m\ddot{x}(t) = -kx(t) \;\; or \;\; m\ddot{x}(t) + kx(t) = 0 \tag{2}$$

where $\ddot{x}(t)$ is the second derivative of the displacement with respect to time.

The solution this periodic motion based on physical observation and experience from watching this mass and spring system is,

$$x(t) = A\sin(\omega_n t + \phi) \tag{3}$$

where A is the amplitude of the displacement, ω_n is the angular natural frequency which determines the interval in time during which the function repeats itself, and ϕ is the phase which determines the initial value of the sine function [2]. Referring to this paper, time t will be measured in seconds (s), the phase in radians (rad) and the angular frequency, ω_n, in radians per seconds (rad/s) which is calculated with the physical properties of the of the mass (m) and spring stiffness (k),

$$\omega_n{}^2 = \frac{k}{m} \quad or \quad \omega_n = \sqrt{\frac{k}{m}} \tag{4}$$

The frequency measures in hertz (Hz) or cycles per seconds (cycles/s) is calculated by,

$$f_n = \frac{\omega_n}{2\pi} \tag{5}$$

By differentiating the equation (3) yields to the velocity which is given by,

$$\dot{x}(t) = \omega_n A\cos(\omega_n t + \phi) \tag{6}$$

differentiating once again yields to the acceleration that is given by,

$$\ddot{x}(t) = -\omega_n{}^2 A\sin(\omega_n t + \phi) \tag{7}$$

The constants of integration, A and ϕ, need to be evaluated with the initial state of motion of system. At time equal to zero (t = 0), t_0, the mass is displaced a distance x_0 and the initial velocity is v_0. Substituting the initial conditions into the equations of motion yields to,

$$x_0 = x(0) = A\sin(\omega_n 0 + \phi) = A\sin(\phi) \tag{8}$$

$$v_0 = \dot{x}(0) = \omega_n A\cos(\omega_n 0 + \phi) = \omega_n A\cos(\phi) \tag{9}$$

VISCOUS DAMPING

Real tests and observations do suggest that the spring-and-mass model equations have to be modified in order to account for the decaying motion that occurs in real life. By adding a term to the equation of motion in the form of,

$$f_c = c\dot{x}(t) \tag{11}$$

where c is called the damping coefficient with units of force per velocity.

By making a force balance on the mass, the equation of motion in the x-direction $x(t)$ yields to,

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{11}$$

In order to solve Equation (11) a particular solution in the form of $x(t) = a\lambda e^{\lambda t}$ is chosen. Substituting this solution into the equation of motion gives,

$$(m\lambda^2 + c\lambda + k)ae^{\lambda t} = 0 \tag{12}$$

since $ae^{\lambda t}$ cannot be equal to zero, then,

$$m\lambda^2 + c\lambda + k = 0 \tag{13}$$

To solve this second order differential equation, the quadratic equation is used in which it yields to two solutions,

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4km} \tag{14}$$

Examination of this expression indicates that the root λ will be real or complex depending on the value of the discriminant, $\sqrt{c^2 - 4km}$ [2]. As long as m, c, and k are positive real numbers, λ_1 and $\lambda_2$will be distinct negative real numbers if $c^2 - 4km > 0$ [2]. If the discriminant is negative, the roots will be a complex conjugate pair with negative real parts [2]. If the discriminant is zero, the two roots λ_1 and λ_2 are equal negative real numbers [2]. For these three cases, the critical damping coefficient, c_{cr}, is defined as,

$$c_{cr} = 2m\omega_n = 2\sqrt{km} \tag{15}$$

The non-dimensional number ζ is called the damping ratio and is defined as,

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\,\omega_n} = \frac{c}{2\sqrt{km}} \tag{16}$$

ROTATING UNBALANCE

Common sources of harmful vibrations are created by rotating machinery. Many machines and devices have rotating components, usually driven by electric components [2]. Some irregularities in the distribution of mass in the rotating component can cause substantial vibration [2]. This is called rotating unbalance [2].

Pictured on Figure (1), the frequency of rotation of the machine is denoted by the ω_r, the unbalance mass as m_0, and the distance from the center of rotation as e. By making a force balance in the vertical direction, the free body diagram yield to,

$$m_0(\ddot{x} + \ddot{x}_r) = -F_r \quad (17)$$

and summing the forces, in the vertical direction, from the free body diagram of the machine yield to,

$$(m - m_0)\ddot{x} = F_r - c\dot{x} - kx \quad (18)$$

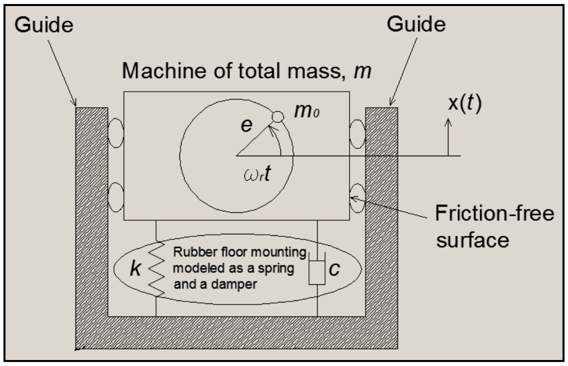


Figure 1
Model of a Machine Causing Support Motion [2]

Combining this two Equations (17) and (18) yields to the equation of motion,

$$m\ddot{x} + m_0\ddot{x}_r + c\dot{x} + kx = 0 \quad (19)$$

By setting $x_r = e\sin(\omega_r t)$ and assuming that the machine is rotating at a constant frequency, ω_r, then,

$$\ddot{x}_r = -e\,\omega_r\sin(\omega_r t) \quad (20)$$

Substituting Equation (20) into Equation (19) the equation of motion yields to,

$$m\ddot{x} + c\dot{x} + kx = m_0\,e\,\omega_r^{\,2}\sin(\omega_r t) \quad (21)$$

Using the particular solution in the form of,

$$x_p(t) = X\cos(\omega_r t + \theta) \quad (22)$$

where r is the frequency ratio denoted by,

$$r = \frac{\omega}{\omega_n} \quad (23)$$

the amplitude of the steady state displacement, X, and the phase angle, θ, are given by,

$$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \quad (24)$$

$$\theta = tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \quad (25)$$

Vibration of a Resiliently Supported Rigid Body

The following sets of equations define the motion of a rigid body on linear massless resilient supporting elements for various degrees of freedom and dynamic excitations. These six equations are solved simultaneously with numerous terms.

System of Coordinates

In this paper, will be discussing the phenomena of a vibrating rigid body on a resilient supported element which is caused by a vibratory force or moment that is generated within the body. The motion of the rigid body is referred to a fixed "inertial" frame of reference represented by a Cartesian coordinate system $\bar{X},\bar{Y},\bar{Z}$ [3]. A similar system of coordinate X,Y,Z fixed in the body has its origin at the center of mass [3]. The translational displacements of the center of mass of the body are x_c,y_c,z_c in the $\bar{X},\bar{Y},\bar{Z}$ direction, respectively and the rotational displacements of the center of mass of the body are characterized by the angles of rotation α,β,γ of the body axes about the in the $\bar{X},\bar{Y},\bar{Z}$ axes respectively [3]. Therefore, the displacement of a point b in the body (with coordinated b_x,b_y,b_z in the X,Y,Z directions, respectively) are the sums of the components of the center of mass displacements in the direction of $\bar{X},\bar{Y},\bar{Z}$ axes plus the tangential components of the rotational displacements of the body [3],

$$x_b = x_c + b_z\beta - b_y\delta \quad (26)$$

$$y_b = y_c - b_z\alpha + b_x\gamma \quad (27)$$

$$z_b = z_c - b_x\beta + b_y\alpha \quad (28)$$

Equations of Small Motion of a Rigid Body

The equations of motion for the translation of a rigid body are given as,

$$m\ddot{x}_c = \boldsymbol{F}_x \quad (29)$$

$$m\ddot{y}_c = \boldsymbol{F}_y \quad (30)$$

$$m\ddot{z}_c = \boldsymbol{F}_z \quad (31)$$

where m is the mass of the the body, $\boldsymbol{F}_x, \boldsymbol{F}_y, \boldsymbol{F}_z$ are the sumation of all the forces acting on the body, and $\ddot{x}_c, \ddot{y}_c, \ddot{z}_c$ are the accelerations of the center of mass in the $\bar{X}, \bar{Y}, \bar{Z}$ direction, respectively [3]. The equations of motion for the rotation of a rigid body are,

$$I_{xx}\ddot{\alpha} - I_{xy}\ddot{\beta} - I_{xz}\ddot{\gamma} = \boldsymbol{M}_x \quad (32)$$

$$-I_{xy}\ddot{\alpha} + I_{yy}\ddot{\beta} - I_{yz}\ddot{\gamma} = \boldsymbol{M}_y \quad (33)$$

$$-I_{xz}\ddot{\alpha} - I_{yz}\ddot{\beta} + I_{zz}\ddot{\gamma} = \boldsymbol{M}_z \quad (34)$$

where $\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}$ are the rotational accelerations about the X, Y, Z axes $\boldsymbol{M}_x, \boldsymbol{M}_y, \boldsymbol{M}_z$ are the summation of torques acting on a rigid body about the $\bar{X}, \bar{Y}, \bar{Z}$ axes, respectively, and $I_{xx}, I_{xy}, I_{xz}, I_{xy}, I_{yy}, I_{yz}, I_{xz}, I_{yz}, I_{zz}$ are the moments and products of inertia of the rigid body [3].

Internal Properties of a Rigid Body

The properties of a rigid body that are significant in dynamics and vibration are the mass (or center of gravity), the moments of inertia, and the direction of the principal inertial axis [3].

Mass

The mass of a body is calculated by integrating the product of mass density $\rho(V)$ and elemental volume dV over the body [3],

$$m = \int_v \rho(V)\, dV \quad (35)$$

Center of Mass

The center of mass (or the center of gravity) is the point located by the vector,

$$r_c = \frac{1}{m}\int_m r(m)\, dm \quad (36)$$

where $r(m)$ is the radius vector of the element of mass dm [3]. The center of mass of the body in a Cartesian coordinate system X, Y, Z is located,

$$X_c = \frac{1}{m}\int_m X(V)\rho(V)\, dV \quad (37)$$

$$Y_c = \frac{1}{m}\int_m Y(V)\rho(V)\, dV \quad (38)$$

$$Z_c = \frac{1}{m}\int_m Z(V)\rho(V)\, dV \quad (39)$$

where $X(V), Y(V), Z(V)$ are the X, Y, Z coordinates of the element volume $dV$and m is the mass of the body [3].

Moment and Product of Inertia

The moments of inertia and products of inertia of a rigid body with respect to the orthogonal axes X, Y, Z fixed to the body are,

$$I_{xx} = \int_m (Y^2 + Z^2)\, dm \quad (40)$$

$$I_{yy} = \int_m (X^2 + Z^2)\, dm \quad (41)$$

$$I_{zz} = \int_m (X^2 + Y^2)\, dm \quad (42)$$

$$I_{xy} = \int_m XY\, dm \quad (43)$$

$$I_{xz} = \int_m XZ\, dm \quad (44)$$

$$I_{yz} = \int_m YZ\, dm \quad (45)$$

where dm is the infinitesimal element of mass located at the coordinates X, Y, Z, and the integration is taken over the mass of the body [3].

Properties of Resilient Supports

A resilient support is considered to be a three dimensional element having two terminals or end connections [3]. When the end connections are moved one relative to the other in any direction, the element resists such motion [3]. Here, the element is considered massless, the force that resists relative motion across the element is considered to consist of a spring force that is directly proportional to the relative displacement (deflection across the element) and a damping force that is directly proportional to the relative velocity (velocity across the element) [3]. Such element is defined as a resilient support [3]. The *principal elastic axes* of the resilient element are those axes that for which the element, when unconstrained, experiences a deflection collinear with the direction of the applied forces [3].

In dynamics, the rigid body sometimes vibrates in modes that are coupled by the properties of the resilient elements well as their location [3]. If a rigid body experiences a static displacement in the x direction, of the X axis only, a resilient element opposes this motion by exerting a force $k_{xx}x$ on the body in the direction of the X axis, where one subscript of the spring constant indicates the direction of the force exerted by the element and the other subscript indicates the direction of the deflection [3]. If the X direction is not a principal elastic direction of the element and the body experiences a static displacement x in the X direction, the body is acted upon by a force $k_{yx}x$ in the Y direction if no displacement y is permitted [3]. The stiffness has reciprocal properties; $k_{yx} = k_{xy}$ [3]. Therefore, the stiffness of a resilient element can be represented pictorially by the combination of three mutually perpendicular, idealized springs oriented along the principal elastic directions of the resilient element [3]. Each spring has stiffness equal to the principal stiffness represented [3].

A resilient element is assumed to have damping properties such that each spring representing a value of principal stiffness is paralleled by an idealized viscous damper, each damper representing a value of principal damping [3]. Hence, coupling through damping exists in a manner similar to coupling through stiffness [3]. Consequently, the viscous damping coefficient c is analogous to the spring coefficient k; i.e., the force exerted by the damping of the resilient element in response to a velocity $\dot{x}$ is $c_{xx}\dot{x}$ the direction of the X axis and $c_{yx}\dot{x}$ in the direction of the Y axis if $\dot{y}$ is zero [3]. Reciprocity exists; i.e., c_{yx}= c_{xy} [3].

The point of intersection of the principal elastic axes of a resilient element is designated as the *elastic center of the resilient element* [3]. The elastic center is important since it defines the theoretical point location of the resilient element for use in the equations of motion of a resiliently supported rigid body [3]. For example, the torque on the rigid body about the Y axis due to a force $k_{xx}x$ transmitted by a resilient element in the X direction is $k_{xx}a_z x$, where a_z is the Z coordinate of the elastic center of the resilient element [3].

Equations of Motion for a Resiliently Supported Rigid Body

The differential equations of motion for the rigid body are given,

$$m\ddot{x}_c = \boldsymbol{F}_x \tag{46}$$

$$m\ddot{y}_c = \boldsymbol{F}_y \tag{47}$$

$$m\ddot{z}_c = \boldsymbol{F}_z \tag{48}$$

$$I_{xx}\ddot{\alpha} - I_{xy}\ddot{\beta} - I_{xz}\ddot{\gamma} = \boldsymbol{M}_x \tag{49}$$

$$-I_{xy}\ddot{\alpha} + I_{yy}\ddot{\beta} - I_{yz}\ddot{\gamma} = \boldsymbol{M}_y \tag{50}$$

$$-I_{xz}\ddot{\alpha} - I_{yz}\ddot{\beta} + I_{zz}\ddot{\gamma} = \boldsymbol{M}_z \tag{51}$$

where the $\boldsymbol{F}$'s and $\boldsymbol{M}$'s represent the forces and moments acting on the body, either directly or through the resilient supporting elements [3]. A rigid body at rest with an inertial set of axes $\bar{X}, \bar{Y}, \bar{Z}$ and a coincident set of axes fixed in the rigid body can have both sets of axes passing through the center-of-mass [3]. A typical resilient element is represented by parallel spring and viscous damper combinations arranged respectively parallel with $\bar{X}, \bar{Y}, \bar{Z}$ axes [3]. Resilient element can also with its principal axes not parallel with $\bar{X}, \bar{Y}, \bar{Z}$ [3].

The displacement of the center-of-gravity of the body in the $\bar{X}, \bar{Y}, \bar{Z}$ directions is indicated by x_c, y_c, z_c, respectively; and rotation of the rigid body about these axes is indicated by α, β, γ, respectively [3]. Each resilient element is represented by three mutually perpendicular spring-damper combinations [3]. One end of each such combination is attached to the rigid body; the other end is considered to be attached to a foundation whose corresponding translational displacement is defined by $u, v, w,$ in the $\bar{X}, \bar{Y}, \bar{Z}$ directions, respectively, and whose rotational displacement about these axes is defined by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, respectively [3]. The point of attachment of each of the idealized resilient elements is located at the coordinate distances a_x, a_y, a_z of the elastic center of the resilient element [3].

Consider the rigid body to experience a translational displacement x_c of its center-of-gravity and no other displacement, and neglect the effects of the viscous dampers [3]. The force developed by a resilient element has the effect of a force $-k_{xx}(x_c - u)$ in the X direction, a moment $k_{xx}(x_c - u)a_y$ in the γ coordinate (about the Z axis), and a moment $-k_{xx}(x_c - u)a_z$ in the β coordinate (about the Y axis) [3]. Furthermore, the coupling stiffness causes a force $-k_{xy}(x_c - u)$ in the Y direction and a force $-k_{xz}(x_c - u)$ in the Z direction [3]. These forces have the moments $k_{xy}(x_c - u)a_z$ in the α coordinate; $-k_{xy}(x_c - u)a_x$ in the γ coordinate; $k_{xz}(x_c - u)a_x$ in the β coordinate; and $-k_{xz}(x_c - u)a_y$ in the α coordinate [3]. By considering in a similar manner the forces and moments developed by a resilient element for successive displacements of the rigid body in the three translational and three rotational coordinates, and summing over the number of resilient elements, the equations of motion are written as follows [3].

$$
\begin{aligned}
m\ddot{x}_c &+ \Sigma c_{xx}(\dot{x}_c - \dot{u}) + \Sigma k_{xx}(x_c - u) \\
&+ \Sigma c_{xy}(\dot{y}_c - \dot{v}) + k_{xy}(y_c - v) + \Sigma c_{xz}(\dot{z}_c - \dot{w}) + \Sigma k_{xz}(z_c - w) \\
&+ \Sigma\left(c_{xz}a_y - c_{xy}a_z\right)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma\left(k_{xz}a_y - k_{xy}a_z\right)(\alpha - \boldsymbol{\alpha}) + \Sigma\left(c_{xx}a_z - c_{xz}a_x\right)(\dot{\beta} - \dot{\boldsymbol{\beta}}) \\
&+ \Sigma\left(k_{xx}a_z - k_{xz}a_x\right)(\beta - \boldsymbol{\beta}) + \Sigma\left(c_{xy}a_x - c_{xx}a_y\right)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) + \Sigma\left(k_{xy}a_x - k_{xx}a_y\right)(\gamma - \boldsymbol{\gamma}) \\
&= F_x
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
I_{xx}\ddot{\alpha} - I_{xy}\ddot{\beta} - I_{xz}\ddot{\gamma} &+ \Sigma\left(c_{xz}a_y - c_{xy}a_z\right)(\dot{x}_c - \dot{u}) + \Sigma\left(k_{xz}a_y - k_{xy}a_z\right)(x_c - u) \\
&+ \Sigma\left(c_{yz}a_y - c_{yy}a_z\right)(\dot{y}_c - \dot{v}) + \Sigma\left(k_{yz}a_y - k_{yy}a_z\right)(y_c - v) \\
&+ \Sigma\left(c_{zz}a_y - c_{yz}a_z\right)(\dot{z}_c - \dot{w}) + \Sigma\left(k_{zz}a_y - k_{yz}a_z\right)(z_c - w) \\
&+ \Sigma\left(c_{yy}{a_z}^2 + c_{zz}{a_y}^2 - 2c_{yz}a_y a_z\right)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma\left(k_{yy}{a_z}^2 + k_{zz}{a_y}^2 - 2k_{yz}a_y a_z\right)(\alpha - \boldsymbol{\alpha}) \\
&+ \Sigma\left(c_{xz}a_y a_z + c_{yz}a_x a_z - c_{zz}a_x a_y - c_{xy}{a_z}^2\right)(\dot{\beta} - \dot{\boldsymbol{\beta}}) \\
&+ \Sigma\left(k_{xz}a_y a_z + k_{yz}a_x a_z - k_{zz}a_x a_y - k_{xy}{a_z}^2\right)(\beta - \boldsymbol{\beta}) \\
&+ \Sigma\left(c_{xy}a_y a_z + c_{yz}a_x a_y - c_{yy}a_x a_z - c_{xz}{a_y}^2\right)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) \\
&+ \Sigma\left(k_{xy}a_y a_z + k_{yz}a_x a_y - k_{yy}a_x a_z - k_{xz}{a_y}^2\right)(\gamma - \boldsymbol{\gamma}) = \boldsymbol{M}_x
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
m\ddot{y}_c &+ \Sigma c_{xy}(\dot{x}_c - \dot{u}) + \Sigma k_{xy}(x_c - u) \\
&+ \Sigma c_{yy}(\dot{y}_c - \dot{v}) + \Sigma k_{yy}(y_c - v) + \Sigma c_{yz}(\dot{z}_c - \dot{w}) + \Sigma k_{yz}(z_c - w) \\
&+ \Sigma\left(c_{yz}a_y - c_{yy}a_z\right)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma\left(k_{yz}a_y - k_{yy}a_z\right)(\alpha - \boldsymbol{\alpha}) + \Sigma\left(c_{xy}a_z - c_{yz}a_x\right)(\dot{\beta} - \dot{\boldsymbol{\beta}}) \\
&+ \Sigma\left(k_{xy}a_z - k_{yz}a_x\right)(\beta - \boldsymbol{\beta}) + \Sigma\left(c_{yy}a_x - c_{xy}a_y\right)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) + \Sigma\left(k_{yy}a_x - k_{xy}a_y\right)(\gamma - \boldsymbol{\gamma}) \\
&= F_y
\end{aligned} \tag{54}
$$

$$-I_{xy}\ddot{\alpha} + I_{yy}\ddot{\beta} - I_{yz}\ddot{\gamma} + \Sigma(c_{xx}a_z - c_{xz}a_x)(\dot{x}_c - \dot{u}) + \Sigma(k_{xx}a_z - k_{xz}a_x)(x_c - u) + \Sigma(c_{xy}a_z - c_{yz}a_x)(\dot{y}_c - \dot{v}) + \Sigma(k_{xy}a_z - k_{yz}a_x)(y_c - v) + \Sigma(c_{xz}a_z - c_{zz}a_x)(\dot{z}_c - \dot{w}) + \Sigma(k_{xz}a_z - k_{zz}a_x)(z_c - w) + \Sigma(c_{xz}a_y a_z + c_{yz}a_x a_z - c_{zz}a_x a_y - c_{xy}a_z{}^2)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma(k_{xz}a_y a_z + k_{yz}a_x a_z - k_{zz}a_x a_y - k_{xy}a_z{}^2)(\alpha - \boldsymbol{\alpha}) + \Sigma(c_{xx}a_z{}^2 + c_{zz}a_x{}^2 - 2c_{xz}a_x a_z)(\dot{\beta} - \dot{\boldsymbol{\beta}}) + \Sigma(k_{xx}a_z{}^2 + k_{zz}a_x{}^2 - 2k_{xz}a_x a_z)(\beta - \boldsymbol{\beta}) + \Sigma(c_{xy}a_x a_z + c_{xz}a_x a_y - c_{xx}a_y a_z - c_{yz}a_x{}^2)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) + \Sigma(k_{xy}a_x a_z + k_{xz}a_x a_y - k_{xx}a_y a_z - k_{yz}a_x{}^2)(\gamma - \boldsymbol{\gamma}) = \boldsymbol{M}_y \quad (55)$$

$$m\ddot{z}_c + \Sigma c_{xz}(\dot{x}_c - \dot{u}) + \Sigma k_{xz}(x_c - u) + \Sigma c_{yz}(\dot{y}_c - \dot{v}) + \Sigma k_{yz}(y_c - v) + \Sigma c_{zz}(\dot{z}_c - \dot{w}) + \Sigma k_{zz}(z_c - w) + \Sigma(c_{zz}a_y - c_{yz}a_z)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma(k_{zz}a_y - k_{yz}a_z)(\alpha - \boldsymbol{\alpha}) + \Sigma(c_{xz}a_z - c_{zz}a_x)(\dot{\beta} - \dot{\boldsymbol{\beta}}) + \Sigma(k_{xz}a_z - k_{zz}a_x)(\beta - \boldsymbol{\beta}) + \Sigma(c_{yz}a_x - c_{xz}a_y)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) + \Sigma(k_{yz}a_x - k_{xz}a_y)(\gamma - \boldsymbol{\gamma}) = F_z \quad (56)$$

$$-I_{xz}\ddot{\alpha} - I_{yz}\ddot{\beta} + I_{zz}\ddot{\gamma} + \Sigma(c_{xy}a_x - c_{xx}a_y)(\dot{x}_c - \dot{u}) + \Sigma(k_{xy}a_x - k_{xx}a_y)(x_c - u) + \Sigma(c_{yy}a_x - c_{xy}a_y)(\dot{y}_c - \dot{v}) + \Sigma(k_{yy}a_x - k_{xy}a_y)(y_c - v) + \Sigma(c_{yz}a_x - c_{xz}a_y)(\dot{z}_c - \dot{w}) + \Sigma(k_{yz}a_x - k_{xz}a_y)(z_c - w) + \Sigma(c_{xy}a_y a_z + c_{yz}a_x a_z - c_{yy}a_x a_z - c_{xz}a_y{}^2)(\dot{\alpha} - \dot{\boldsymbol{\alpha}}) + \Sigma(k_{xy}a_y a_z + k_{yz}a_x a_z - k_{yy}a_x a_z - k_{xz}a_y{}^2)(\alpha - \boldsymbol{\alpha}) + \Sigma(c_{xy}a_x a_y + c_{yz}a_x a_z - c_{xx}a_y a_z - c_{yz}a_x{}^2)(\dot{\beta} - \dot{\boldsymbol{\beta}}) + \Sigma(k_{xy}a_x a_y + k_{yz}a_x a_z - k_{xx}a_y a_z - k_{yz}a_x{}^2)(\beta - \boldsymbol{\beta}) + \Sigma(c_{xx}a_y{}^2 + c_{yy}a_x{}^2 - 2c_{xy}a_x a_y)(\dot{\gamma} - \dot{\boldsymbol{\gamma}}) + \Sigma(k_{xx}a_y{}^2 + k_{yy}a_x{}^2 - 2k_{xy}a_x a_y)(\gamma - \boldsymbol{\gamma}) = \boldsymbol{M}_z \quad (57)$$

where the moment and product of inertia are defined by,

$$I_{xx} = \int_m (Y^2 + Z^2)\, dm \quad (58)$$

$$I_{yy} = \int_m (X^2 + Z^2)\, dm \quad (59)$$

$$I_{zz} = \int_m (X^2 + Y^2)\, dm \quad (60)$$

$$I_{xy} = \int_m XY\, dm \quad (61)$$

$$I_{xz} = \int_m XZ\, dm \quad (62)$$

$$I_{yz} = \int_m YZ\, dm \quad (63)$$

the damping coefficient are defined by,

$$c_{xx} = c_p\lambda_{xp}{}^2 + c_q\lambda_{xq}{}^2 + c_r\lambda_{xr}{}^2 \quad (64)$$

$$c_{yy} = c_p\lambda_{yp}{}^2 + c_q\lambda_{yq}{}^2 + c_r\lambda_{yr}{}^2 \quad (65)$$

$$c_{zz} = c_p\lambda_{zp}{}^2 + c_q\lambda_{zq}{}^2 + c_r\lambda_{zr}{}^2 \quad (66)$$

$$c_{xy} = c_p\lambda_{xp}\lambda_{yp} + c_q\lambda_{xq}\lambda_{yq} + c_r\lambda_{xr}\lambda_{yr} \quad (67)$$

$$c_{xz} = c_p\lambda_{xp}\lambda_{zp} + c_q\lambda_{xq}\lambda_{zq} + c_r\lambda_{xr}\lambda_{zr} \quad (68)$$

$$c_{yz} = c_p\lambda_{yp}\lambda_{zp} + c_q\lambda_{yq}\lambda_{zq} + c_r\lambda_{yr}\lambda_{zr} \quad (69)$$

the stiffness coefficient are defined by,

$$k_{xx} = k_p\lambda_{xp}{}^2 + k_q\lambda_{xq}{}^2 + k_r\lambda_{xr}{}^2 \quad (70)$$

$$k_{yy} = k_p\lambda_{yp}{}^2 + k_q\lambda_{yq}{}^2 + k_r\lambda_{yr}{}^2 \quad (71)$$

$$k_{zz} = k_p\lambda_{zp}{}^2 + k_q\lambda_{zq}{}^2 + k_r\lambda_{zr}{}^2 \quad (72)$$

$$k_{xy} = k_p\lambda_{xp}\lambda_{yp} + k_q\lambda_{xq}\lambda_{yq} + k_r\lambda_{xr}\lambda_{yr} \quad (73)$$

$$k_{xz} = k_p\lambda_{xp}\lambda_{zp} + k_q\lambda_{xq}\lambda_{zq} + k_r\lambda_{xr}\lambda_{zr} \quad (74)$$

$$k_{yz} = k_p\lambda_{yp}\lambda_{zp} + k_q\lambda_{yq}\lambda_{zq} + k_r\lambda_{yr}\lambda_{zr} \quad (75)$$

where λ is the cosine angle between the principal elastic axes of the resilient supporting elements and the coordinate's axes [3].

Modal Analysis of the Forced Response Results

The force response of a multiple degree of freedom system can be calculated by using modal analysis. The equation of motion takes the form,

$$[M]\begin{bmatrix}\ddot{x}(t)\\ \ddot{y}(t)\\ \ddot{z}(t)\\ \ddot{\alpha}(t)\\ \ddot{\beta}(t)\\ \ddot{\gamma}(t)\end{bmatrix} + [C]\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\\ \dot{\alpha}(t)\\ \dot{\beta}(t)\\ \dot{\gamma}(t)\end{bmatrix} + [K]\begin{bmatrix}x(t)\\ y(t)\\ z(t)\\ \alpha(t)\\ \beta(t)\\ \gamma(t)\end{bmatrix} = B\begin{bmatrix}F_x(t)\\ F_y(t)\\ F_z(t)\\ M_x(t)\\ M_y(t)\\ M_z(t)\end{bmatrix} \quad (76)$$

where M is the mass matrix, C is the damping matrix and K is the stiffness matrix, B is identity matrix. The modal analysis uses transformation to reduce the equation of motion to decoupled modal equations then they are solves for the individual force response in the modal coordinate system, and then they are transformed back to the physical coordinates system.

In order to solve these equations the modal analysis method was used. First the matrix of eigenvectors P was calculated to decouple the equations of vibrations into six separate equations. The matrices P and $M^{-\frac{1}{2}}$ are used to transformed the vibration problem between two different coordinate systems. These procedures is called modal analysis, because the transformation $S = M^{-\frac{1}{2}}P$, is called the matrix of mode shapes where each column is a mode shape vector, often called the modal matrix is related to the mode shapes of the vibrating systems. The matrix $\tilde{K} = M^{-\frac{1}{2}}KM^{-\frac{1}{2}}$ is called the mass-normalized stiffness, and $\tilde{C} = M^{-\frac{1}{2}}CM^{-\frac{1}{2}}$ mass-normalized damping matrix.

By letting $\boldsymbol{x}(t) = M^{-\frac{1}{2}}\boldsymbol{q}(t)$, and multiplying the x term of Equation (76) by $M^{-\frac{1}{2}}$ yields to,

$$[I]\ddot{q}(t) + [\tilde{C}]\dot{q}(t) + [\tilde{K}]q(t) = M^{-\frac{1}{2}}B\boldsymbol{F}(t) \quad (77)$$

Then, let $q(t) = \mathrm{P}r(t)$, where P is the matrix of eigenvectors of $\tilde{K}$. By multiplying this equation by $M^{-\frac{1}{2}}$ yields to,

$$\ddot{r}(t) + diag[2\zeta\omega]\dot{r}(t) + \Lambda r(t) = P^T M^{-\frac{1}{2}}B\boldsymbol{F}(t) \quad (78)$$

The decoupled modal equation takes the form of,

$$\ddot{r}(t) + 2\zeta\omega\dot{r}(t) + \omega^2 r(t) = f(t) \quad (79)$$

The solution is then calculated by multiplying $x(t) = \mathrm{S}\,\mathrm{r}(t)$. The same procedure is performed with $y(t), z(t), \alpha(t), \beta(t), \gamma(t)$ for our 6 degree-of-freedom system.

Simulation Program and Results

As previously mentioned, the purpose of this project was to create a Matlab© [4] program that could generate the vibration motion and properties of any AHU that is to be design before it is installed. A Matlab© [4] program was created and its results were validated with a more sophisticates simulation program. For this project, ANSYS Workbench© R15 [5] was selected for such a task.

Two sets of springs were selected for in order to validate the program results. For the first simulation run, a set of four (4) Blue springs with a stiffness constant of 46 lb/in. where installed and for the second run a set of four (4) Brown springs with a stiffness constant of 133 lb/in were installed in the model in order to compare their results. A sample of both sets of springs is showed on Figure (2). Both of these springs have a height of 2.75 inches and an outside diameter of 2.0 inches.

For the Blue spring simulation, a motor speed of 600 RPM was selected. The displacement versus time response for the Blue springs using the Matlab© [4] program is shown in Figure (3a) which generated a displacement amplitude of 0.04546 inches. The moments generated on the simulated model obtained in the Matlab© [4] program are shown on Figure (3b). The amplitude versus frequency response using the ANSYS Workbench© [5] program is shown in Figure (4), which generated a displacement amplitude of 0.0476 inches. Their results are compared on Table (1).

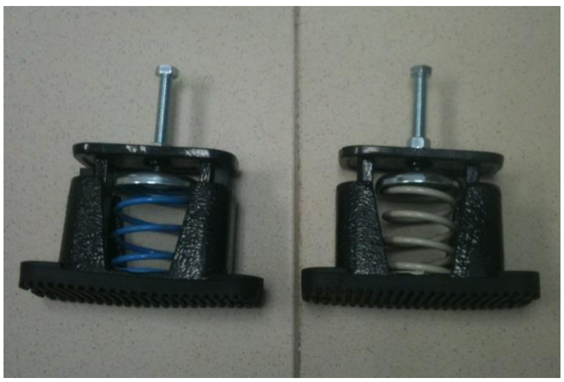
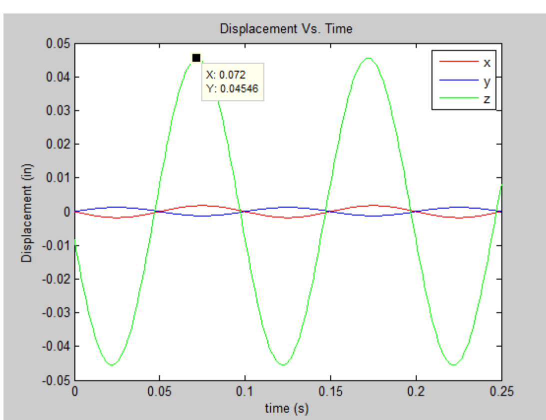


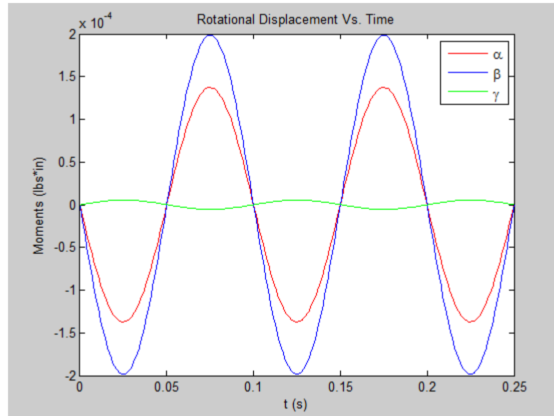
Figure 2

Blue springs and Brown springs used in the Model. [6]

Comparing the Matlab© and ANSYS Workbench© results with the theoretical results, the magnitudes of the displacement distance are very similar. The error percentage for ANSYS Workbench© is very small compared to the Matlab© simulation, although the Matlab© simulation errors can be accredited to the decimal approximation within the program.



(a)



(b)

Figure 3

Displacement versus Time (a) and Moments versus Time (b) for the Blue Springs at a Speed of 600 rpm.

The natural frequency on the Matlab© program gave the same result as the theoretical formula. With the ANSYS Workbench© program, the error is very small which could be attributed to the weight approximation of the simulated AHU model.

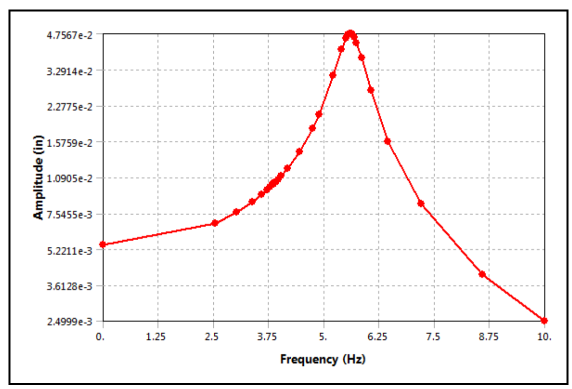


Figure 4

Frequency Response in the z-Axis for the Blue Springs

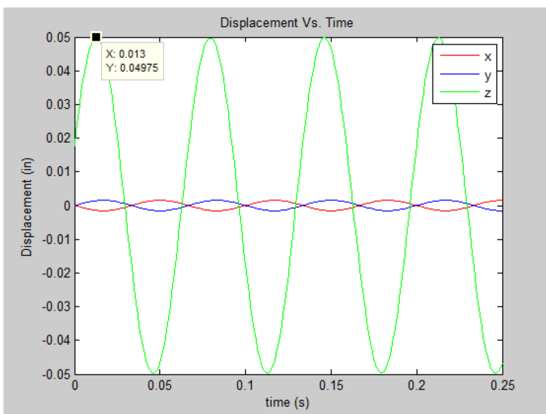
	Blue Springs Results				
	RPM	Wr (rad/s)	r	K (lb/in)	DR
	600.00	62.83	1.78	46	0.0573
	Theory	ANSYS	%Error	MATLAB	%Error
Max Amplitude (in)	0.0479	0.0476	0.64	0.04546	5.04
Natural Frequency (rad/s)	35.27	35.08	0.54	35.27	0.00

Table 1

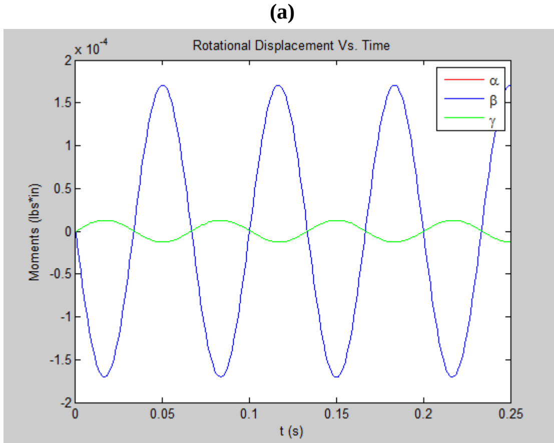
Blue Springs Results.

For the case of the Brown spring simulation, a motor speed of 900 RPM was selected. The displacement versus time response using the Matlab© [4] program is shown in Figure (5a) which generated a displacement amplitude of 0.04975 inches. The moments generated on the simulated model obtained in the Matlab© [4] program using the Brown springs are shown on Figure (5b). The amplitude versus frequency response using the ANSYS Workbench© program is shown in Figure (6), which generated a displacement amplitude of 0.046898 inches. Their results are compared on Table (2).

Once again, comparing the Matlab© [4] and ANSYS Workbench© [5] results with the theoretical results retrieve very similar results. The error percentage for ANSYS Workbench© [5] and the Matlab© [4] program simulation are very similar. The error percentage can be credited to decimal approximation in the program.



(a)



(b)

Figure 5

Displacement versus Time (a) and Moments versus Time (b) for the Brown Springs at a Speed of 900 rpm.

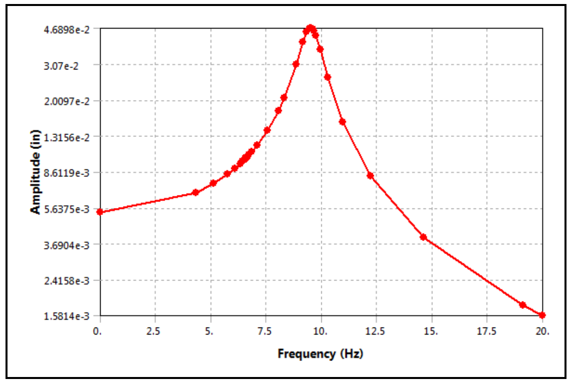


Figure 6

Frequency response in the z-axis for the Brown springs.

The natural frequency results from the Matlab© [4] program gave the same result as the theoretical formula once again. The ANSYS Workbench© [5] program results gave a very small error when compared with the theoretical formula which can be attributed to the weight approximation of the simulated AHU model once again. Nevertheless, the Matlab© [4] program gave excellent results.

	Brown Springs Results				
	RPM	Wr (rad/s)	r	K (lb/in)	DR
	900.00	94.25	1.57	133	0.0890
	Theory	ANSYS	%Error	MATLAB	%Error
Max Amplitude (in)	0.0473	0.0469	0.83	0.0498	5.20
Natural Frequency (rad/s)	59.97	59.84	0.23	59.97	0.00

Table 2

Brown sSprings Results.

CONCLUSIONS

By simulating both, the Blue and Brown spring systems using the Matlab© [4] computer program and validating their results using the ANSYS Workbench© [5] program, we can conclude that the Blue springs, at a motor speed of 600 RPM, absorbs about the same vibrating energy as does the Brown springs at a motor speed of 900 RPM. By simulating the same mass with different spring constants, different damping ratios and different motor speeds we can conclude that the Matlab© [4] program is a useful tool for designers and installers to use in the selection of vibration isolators for AHU.

For future work, this program can be expanded to simulate more complex models. It can also be used to incorporate industrial size chillers, pumps, fans, enthalpy wheels, etc. and any number of combination of system that are used in today's HVAC industry.

ANSYS Workbench© [4] was a great tool to use in order to validate the results of the Matlab© [4] program. Although, the ANSYS Workbench© [5] program is a very useful and expensive tool, the same results can be obtained using a least expensive computer program such as Matlab© [4] which can give very similar results to the end user. This program can be used to predict the behavior and properties of any designed model before the actual and final AHU is built and installed saving time, energy, and most important, money.

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